Bull. Soc. Math. France
147 (3), 2019, p. 443-453

SECONDARY CHARACTERISTIC CLASSES OF LIE ALGEBRA EXTENSIONS

by Stefan Wagner

Abstract. — We introduce the notion of secondary characteristic classes of Lie algebra extensions. As an application of our construction we obtain a new proof of Lecomte's generalization of the Chern-Weil homomorphism.

Résumé (*Classes caractéristique secondaries d'extensions d'algèbre de Lie*). — Nous introduisons une notion de classes caractéristiques secondaires d'extensions d'algèbre de Lie. Comme une application nous obtenons une nouvelle preuve de la généralisation par Lecomte de l'homomorphisme de Chern-Weil.

1. Introduction

Characteristic classes are topological invariants of principal bundles and vector bundles associated with principal bundles. The theory of characteristic classes was started in the 1930s by Stiefel and Whitney. Stiefel studied certain homology classes of the tangent bundle TM of a smooth manifold M, while Whitney considered the case of an arbitrary sphere bundle and introduced the concept of a characteristic cohomology class. In the next decade, Pontryagin constructed important new characteristic classes by studying the homology of real Grassman manifolds and Chern defined characteristic classes for complex vector bundles. Nowadays, characteristic classes are an important tool for many disciplines such as analysis, geometry and modern physics. For example, they

Texte reçu le 16 août 2018, accepté le 25 octobre 2018.

Stefan Wagner, Blekinge Tekniska Högskola • *E-mail :* stefan.wagner@bth.se

Mathematical subject classification (2010). — 17B56, 53A55, 18A05.

Key words and phrases. — secondary characteristic class, Lie algebra extension.

doi:10.24033/bsmf.2788

provide a way to measure the non-triviality of a principal bundle respectively, and the non-triviality of an associated vector bundle. The Chern-Weil homomorphism of a principal bundle is an algebra homomorphism from the algebra of polynomials invariant under the adjoint action of a Lie group G on the corresponding Lie algebra $\mathfrak{g}$, into the even De Rham cohomology $H^{2\bullet}_{\mathrm{dR}}(M,\mathbb{K})$ of the base space M of a principal bundle P with structure group G. This map is achieved by evaluating an invariant polynom f of degree k on the curvature Ω of a connection ω on P and thus obtaining a closed form on the base.

In [4] Lecomte described a cohomological construction which generalizes the classical Chern-Weil homomorphism. In fact, Lecomte's construction associates characteristic classes to every Lie algebra extension and the classical construction of Chern and Weil arises in this context from the Lie algebra extension known as the *Atiyah sequence* of a principal bundle.

In the 1970s, another set of characteristic classes called the *secondary characteristic classes* has been discovered. These classes are also global invariants of principal bundles and are derived in a similar way as the primary characteristic classes from the curvature of adequate connection 1-forms. Secondary characteristic classes appear for example in the Lagrangian formulation of modern quantum field theories and the most well-known might be the so-called *Chern-Simons classes* (see e. g. [2, 1, 5] and references therein).

In this note we investigate secondary characteristic classes in the context of Lie algebra extensions which also provides a new proof of Lecomte's generalization of the Chern-Weil homomorphism. Moreover, we would like to point out that our construction may be used to associate characteristic classes to split Lie algebra extensions, that is, to semidirect products of Lie algebras, because in this situation the primary characteristic classes vanish. From this perspective our results nicely complement the Lie algebraic theory of characteristic classes. This paper is organized as follows.

After this introduction and some preliminaries, we pave the way in Section 3 for our main results and we give a short overview over Lecomte's generalization of the Chern-Weil homomorphism. Section 4 is finally devoted to our main purpose of investigating secondary characteristic classes of Lie algebra extensions. To be more precise, we first introduce the so-called *Bott-Lecomte* homomorphism and show that this map, in fact, gives rise to classes in Lie algebra cohomology (Theorem 4.1). As an application of our construction we obtain a new proof of Lecomte's generalization of the Chern-Weil homomorphism (Corollary 4.2 and Corollary 4.3). Finally, we introduce a notion of secondary characteristic classes of Lie algebra extensions and conclude with discussing characteristic classes of the oscillator algebra.

2. Preliminaries and notations

In this section we provide the most important definitions and notations of Lie algebra cohomology which are repeatedly used in this article. For a detailed background on Lie algebra cohomology we refer, for example, to [3, 1, 6].

Alternating and symmetric maps. — Let V, W be vector spaces and $p \in \mathbb{N}_0$. We call a p-linear map $f : W^p \to V$ *alternating* if $f(w_{\sigma(1)}, \ldots, w_{\sigma(p)}) = \operatorname{sgn}(\sigma) \cdot f(w_1, \ldots, w_p)$ for all $w_1, \ldots, w_p \in W$ and all permutations $\sigma \in S_p$. Furthermore, we write $\operatorname{Alt}^p(W, V)$ for the space of alternating p-linear maps $W^p \to V$. Given a p-linear map $f : W^p \to V$, it is easily checked that

$$\operatorname{Alt}(f) := \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) \cdot f^\sigma \in \operatorname{Alt}^p(W, V),$$

where $f^\sigma(w_1, \ldots, w_p) := f(w_{\sigma(1)}, \ldots, w_{\sigma(p)})$. The *symmetric* p-linear maps $W^p \to V$ are defined accordingly and denoted by $\operatorname{Sym}^p(W, V)$. We will sometimes consider an element $f \in \operatorname{Sym}^p(W, V)$ as a linear map on the *symmetric tensor product* $S^p(W)$, that is, as a map $\tilde{f} : S^p(W) \to V$ such that $\tilde{f}(w_1 \otimes_s \cdots \otimes_s w_p) = f(w_1, \ldots, w_p)$ for all $w_1, \ldots, w_p \in W$.

The wedge product. — Suppose that $\mathfrak{g}$ and V_i, $i = 1, 2, 3$, are vector spaces and that $m : V_1 \times V_2 \to V_3$, $(v_1, v_2) \mapsto v_1 \cdot_m v_2$ is a bilinear map. For $\alpha \in \operatorname{Alt}^p(\mathfrak{g}, V_1)$ and $\beta \in \operatorname{Alt}^q(\mathfrak{g}, V_2)$ we define the *wedge product* $\alpha \wedge_m \beta \in \operatorname{Alt}^{p+q}(\mathfrak{g}, V_3)$ by putting

$$\alpha \wedge_m \beta := \frac{1}{p!q!} \operatorname{Alt}(\alpha \cdot_m \beta),$$

where $(\alpha \cdot_m \beta)(x_1, \ldots, x_{p+q}) := \alpha(x_1, \ldots, x_p) \cdot_m \beta(x_{p+1}, \ldots, x_{p+q})$. The wedge products $\wedge_{\otimes_s}$ induced by the canonical multiplications $S^p(\mathfrak{g}) \times S^q(\mathfrak{g}) \to S^{p+q}(\mathfrak{g})$, $(x, y) \mapsto x \otimes_s y$ will be of particular interest to us.

The Chevalley-Eilenberg complex. — Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. For $p \in \mathbb{N}_0$ we denote the space of alternating p-linear mappings $\mathfrak{g}^p \to V$ by $C^p(\mathfrak{g}, V) := \operatorname{Alt}^p(\mathfrak{g}, V)$ and call its elements *p-cochains*. We use the convention $C^0(\mathfrak{g}, V) = V$ and observe that $C^1(\mathfrak{g}, V) = \operatorname{Lin}(\mathfrak{g}, V)$ is the space of linear maps $\mathfrak{g} \to V$. We also consider

$$C^\bullet(\mathfrak{g}, V) := \bigoplus_{p=0}^{\infty} C^p(\mathfrak{g}, V).$$

On each $C^p(\mathfrak{g},V)$ we define the *Chevalley-Eilenberg differential* $= d_{\mathfrak{g}}$ by

$$d_{\mathfrak{g}}\alpha(x_0,\dots,x_p) := \sum_{j=0}^{p}(-1)^j x_j.\alpha(x_0,\dots,\widehat{x}_j,\dots,x_p) + \sum_{i<j}(-1)^{i+j}\alpha([x_i,x_j],x_0,\dots,\widehat{x}_i,\dots,\widehat{x}_j,\dots,x_p),$$

where $\widehat{x}_j$ means that x_j is omitted. Observe that the right hand side defines for each $\alpha \in C^p(\mathfrak{g},V)$ an element of $C^{p+1}(\mathfrak{g},V)$ because it is alternating. Putting all differentials together, we obtain a linear map $d_{\mathfrak{g}} : C^\bullet(\mathfrak{g},V) \to C^\bullet(\mathfrak{g},V)$. The elements of the subspace $Z^p(\mathfrak{g},V) := \ker(d_{\mathfrak{g}|C^p(\mathfrak{g},V)})$ are called *p-cocycles*, and the elements of the spaces

$$B^p(\mathfrak{g},V) := d_{\mathfrak{g}}(C^{p-1}(\mathfrak{g},V)) \quad \text{and} \quad B^0(\mathfrak{g},V) := \{0\}$$

are called *p-coboundaries*. It is easily checked that $d_{\mathfrak{g}}^2 = 0$ which implies that $B^p(\mathfrak{g},V) \subseteq Z^p(\mathfrak{g},V)$, so that it makes sense to define the p^{th}-*cohomology space of* $\mathfrak{g}$ *with values in the* $\mathfrak{g}$-*module* V, that is, $H^p(\mathfrak{g},V) := Z^p(\mathfrak{g},V)/B^p(\mathfrak{g},V)$.

Covariant derivatives and curvature. — Let $\mathfrak{g}$ be a Lie algebra and V a vector space, considered as a trivial $\mathfrak{g}$-module. Given a linear map $S : \mathfrak{g} \to \mathrm{End}(V)$ and $p \in \mathbb{N}_0$, we write $d_S : C^p(\mathfrak{g},V) \to C^{p+1}(\mathfrak{g},V)$ for the corresponding *covariant derivative* defined by

$$d_S\alpha(x_0,\dots,x_p) := \sum_{j=0}^{p}(-1)^j S(x_j).\alpha(x_0,\dots,\widehat{x}_j,\dots,x_p) + \sum_{i<j}(-1)^{i+j}\alpha([x_i,x_j],x_0,\dots,\widehat{x}_i,\dots,\widehat{x}_j,\dots,x_p).$$

The formula shows that if S is a Lie algebra homomorphism, then d_S is the Lie algebra differential corresponding to the $\mathfrak{g}$-module structure on V given by S. Moreover, given a bilinear map $m : V \times V \to V$ and a linear map $S : \mathfrak{g} \to \mathrm{der}(V,m)$, we have for $\alpha \in C^p(\mathfrak{g},V)$ and $\beta \in C^q(\mathfrak{g},V)$ the relation

$$d_S(\alpha \wedge_m \beta) = d_S\alpha \wedge_m \beta + (-1)^p \alpha \wedge_m d_S\beta. \tag{1}$$

If, finally, V is also a Lie algebra and $\sigma \in C^1(\mathfrak{g},V)$, we denote by $R_\sigma \in C^2(\mathfrak{g},V)$ the corresponding *curvature* which is defined by $R_\sigma(x,y) = [\sigma(x),\sigma(y)] - \sigma([x,y])$. It follows from [6, Prop. I.8] that R_σ satisfies the *abstract Bianchi identity* $d_S R_\sigma = 0$, where $S := \mathrm{ad} \circ \sigma$. That is,

$$\sum_{\text{cyc.}}[\sigma(x),R_\sigma(y,z)] - R_\sigma([x,y],z) = 0 \qquad \forall x,y,z \in \mathfrak{g}.$$

3. Lecomte's generalization of the Chern-Weil map

In this short section we pave the way for our main result and we give a short overview over Lecomte's generalization of the Chern-Weil homomorphism. Indeed, let

$$0 \longrightarrow \mathfrak{n} \longrightarrow \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \longrightarrow 0$$

be a Lie algebra extension and V a $\mathfrak{g}$-module which we also consider as a $\widehat{\mathfrak{g}}$-module with respect to the action $x.v := q(x).v$ for $x \in \widehat{\mathfrak{g}}$ and $v \in V$. Furthermore, let $\sigma : \mathfrak{g} \to \widehat{\mathfrak{g}}$ be a linear section of q and $R_\sigma \in C^2(\mathfrak{g}, \mathfrak{n})$ the corresponding curvature. Given $p \in \mathbb{N}_0$ and $f \in \mathrm{Sym}^p(\mathfrak{n}, V)$, we put

$$f_\sigma = \tilde{f} \circ (R_\sigma \wedge_{\otimes_s} \cdots \wedge_{\otimes_s} R_\sigma) \in C^{2p}(\mathfrak{g}, V)$$

(cf. Section 2). Moreover, we write $\mathrm{Sym}^p(\mathfrak{n}, V)^{\widehat{\mathfrak{g}}}$ for the set of $\widehat{\mathfrak{g}}$-invariant symmetric p-linear maps, that is, the set

$$\left\{ f \in \mathrm{Sym}^p(\mathfrak{n}, V) \,|\, x.f(y_1, \ldots, y_p) = \sum_{i=1}^{p} f(y_1, \ldots, S(x)y_i, \ldots, y_p) \qquad \forall x \in \mathfrak{g} \right\},$$

where $S : \mathfrak{g} \to \mathrm{der}(\mathfrak{n})$ denotes the linear map defined by $S(x) := \mathrm{ad}(\sigma(x))$. Lecomte's generalization of the Chern-Weil map then reads as follows:

THEOREM 3.1. — (Lecomte [4, Thm. 2.3])

(a) For each $p \in \mathbb{N}_0$, there is a natural map

$$C_p : \mathrm{Sym}^p(\mathfrak{n}, V)^{\widehat{\mathfrak{g}}} \to H^{2p}(\mathfrak{g}, V), \quad f \mapsto \frac{1}{p!}[f_\sigma],$$

which is independent of the choice of the section σ.

(b) Suppose, in addition, that $m_V : V \times V \to V$ is an associative multiplication and that $\mathfrak{g}$ acts on V by derivations, i.e., m_V is $\mathfrak{g}$-invariant. Then $(C^\bullet(\mathfrak{g}, V), \wedge_{m_V})$ is an associative algebra, inducing an algebra structure on $H^\bullet(\mathfrak{g}, V)$. Further, $(\mathrm{Sym}^\bullet(\mathfrak{n}, V), \vee_{m_V})$ is an associative algebra, and the maps $(C_p)_{p \in \mathbb{N}_0}$ combine to an algebra homomorphism

$$C : \mathrm{Sym}^\bullet(\mathfrak{n}, V)^{\widehat{\mathfrak{g}}} \to H^{2\bullet}(\mathfrak{g}, V).$$

4. Secondary characteristic classes of Lie algebra extensions

In this section we finally introduce a notion of secondary characteristic classes of Lie algebra extensions. As a spin-off of our construction we obtain another proof for Lecomte's generalization of the Chern-Weil homomorphism.