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J-INVARIANT OF LINEAR ALGEBRAIC GROUPS

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ABSTRACT. – Let G be a semisimple linear algebraic group of inner type over a field F, and let X be a projective homogeneous G-variety such that G splits over the function field of X. We introduce the J-invariant of G which characterizes the motivic behavior of X, and generalizes the J-invariant defined by A. Vishik in the context of quadratic forms.

We use this J-invariant to provide motivic decompositions of all generically split projective homogeneous G-varieties, e.g. Severi-Brauer varieties, Pfister quadrics, maximal orthogonal Grassmannians, varieties of Borel subgroups of G. We also discuss relations with torsion indices, canonical dimensions and cohomological invariants of the group G.

RÉSUMÉ. – Soit G un groupe algébrique linéaire semi-simple de type intérieur sur un corps F et soit X un G-espace homogène projectif tel que le groupe G soit déployé sur le point générique de X. Nous introduisons le J-invariant de G qui caractérise le comportement motivique de X et généralise le J-invariant défini par A. Vishik dans le cadre des formes quadratiques.

Nous utilisons cet invariant pour obtenir les décompositions motiviques de tous les G-espaces homogènes projectifs qui sont génériquement déployés, par exemple les variétés de Severi-Brauer, les quadriques de Pfister, la grassmannienne des sous-espaces totalement isotropes maximaux d'une forme quadratique, la variété des sous-groupes de Borel de G. Nous discutons également les relations avec les indices de torsion, la dimension canonique et les invariants cohomologiques du groupe G.

Introduction

Let G be a semisimple linear algebraic group over a field F, and let X be a projective homogeneous G-variety. We are interested in direct sum decomposition of the Grothendieck-Chow motive $\mathcal{M}(X)$ of X.

Motivic decompositions are fundamental mathematical tools which in recent years have led to the resolution of several classical problems. For instance, the motivic decomposition of a Pfister quadric plays a major role in Voevodsky's proof of the Milnor conjecture. A proof of the generalization of this conjecture known as the Bloch-Kato conjecture has been announced by Rost and Voevodsky, and uses in an essential way the motivic decompositions of norm varieties which are closely related to projective homogeneous varieties.

As another application of motivic decompositions we mention Vishik's [36] recent breakthrough concerning the Kaplansky problem, where Vishik uses the notion of a J-invariant of an orthogonal group. In fact, our paper has been motivated by Vishik's work, and the J-invariant we introduce is a generalization to an arbitrary semisimple algebraic group.

Concerning decompositions of motives of homogenous G-varieties X it was first observed by Köck [25] that if the group G is split (i.e. contains a split maximal torus), the motive of X is isomorphic to a direct sum of twisted Tate motives, thus has the simplest possible decomposition. Chernousov-Gille-Merkurjev [7] and Brosnan [4] proved that if G is isotropic (i.e. contains a split 1-dimensional torus), the motive of X decomposes as a direct sum of motives of projective homogeneous varieties of smaller dimensions corresponding to anisotropic groups, which reduces the problem to the anisotropic case. For anisotropic groups only a few partial results are known. In this case the components of a motivic decomposition of X are generally expected to be of a non-geometric nature, i.e. not the (twisted) motives of some other varieties. The first examples of such decompositions were provided by Rost [32] who proved that the motive of a Pfister quadric decomposes as a direct sum of twisted copies of a so-called Rost motive $\mathcal{R}$, which is a priori non-geometric. The motives of Severi-Brauer varieties have been computed by Karpenko [21]. For examples of motivic decompositions of exceptional varieties, see Bonnet [2] (varieties of type G_2), and Nikolenko-Semenov-Zainoulline [30] (varieties of type F_4). Note that in all these examples the group G splits over the generic point of X; we will call such varieties *generically split*.

Our main result is a uniform proof of the above results. We show (see Theorem 5.17):

MAIN THEOREM. – *Let G be a semisimple linear algebraic group of inner type over a field F and let p be a prime integer. Let X be a generically split projective homogeneous G-variety. Then the Chow motive of X with $\mathbb{Z}/p$-coefficients is isomorphic to a direct sum*

$$\mathcal{M}(X;\mathbb{Z}/p) \simeq \bigoplus_{i\in\mathcal{I}} \mathcal{R}_p(G)(i)$$

of twisted copies of an indecomposable motive $\mathcal{R}_p(G)$ for some finite multiset $\mathcal{I}$ of non-negative integers.

Observe that the motive $\mathcal{R}_p(G)$ depends on G and p but not on the type of a parabolic subgroup defining X. Moreover, considered with $\mathbb{Q}$-coefficients $\mathcal{R}_p(G)$ always splits as a direct sum of twisted Tate motives.

Our proof has two key ingredients. The first is the *Rost Nilpotence Theorem* proved for projective quadrics by Rost, and generalized to arbitrary projective homogeneous varieties by Brosnan [4] and Chernousov-Gille-Merkurjev [7]. Roughly speaking, this result plays the role of the Galois descent for motivic decompositions over a separable closure $\bar{F}$ of F, and reduces the problem to the *description of idempotent cycles* in the endomorphism group $\mathrm{End}(\mathcal{M}(X_{\bar{F}};\mathbb{Z}/p))$ which are defined over F. The second key point in our proof comes from the topology of compact Lie groups. In [20] Kac invented the notion of *p-exceptional degrees*, which are numbers that relate the degrees of mod p basic polynomial invariants with the p-torsion part of the Chow ring of a compact Lie group. Zainoulline [38] has shown that these numbers describe the generic part of the subgroup of cycles in $\operatorname{End}(\mathcal{M}(X_{\bar{F}};\mathbb{Z}/p))$ defined over F.

To describe the whole subgroup we introduce the *J-invariant* of a group G mod p, which is a list of non-negative integers denoted by $J_p(G)$ (see Definition 4.6). In most cases the values of $J_p(G)$ are implicitly computed in [20] and can easily be extracted from Table 4.13. The J-invariant measures the 'size' of the motive $\mathcal{R}_p(G)$ and in this way characterizes the motivic decomposition of X. For instance, if the J-invariant takes its minimal possible non-trivial value $J_p(G)=(1)$, then the motive $\mathcal{R}_p(G)\otimes\mathbb{Q}$ has the following recognizable decomposition (cf. [37, § 5] and [33, § 5])

$$\mathcal{R}_p(G)\otimes\mathbb{Q}\simeq\bigoplus_{i=0}^{p-1}\mathbb{Q}(i\cdot\tfrac{p^{n-1}-1}{p-1}),\ \text{where } n=2 \text{ or } 3.$$

The assignment $G\mapsto\mathcal{R}_p(G)$ can be viewed as a motivic analogue of the *cohomological invariant* of G given by the Tits class of G if $n=2$, and by the Rost invariant of G if $n=3$. In these cases the motive $\mathcal{R}_p(G)$ coincides with a *generalized Rost motive*.

We also generalize some of the results in [5]. Using the motivic version of the main result of Edidin-Graham [12] on *cellular fibrations* we provide a general formula which expresses the motive of the total space of a cellular fibration in terms of the motives of its base (see Theorem 3.7). We also provide several criteria for the existence of *liftings of motivic decompositions* via the reduction map $\mathbb{Z}\to\mathbb{Z}/m$, and prove such liftings always exist (see Theorem 2.16).

The paper is organized as follows. In the first section we provide several auxiliary facts concerning motives and rational cycles. The rather technical Section 2 is devoted to lifting of idempotents. In Section 3 we discuss the motives of cellular fibrations. The next section is devoted to the notion of a J-invariant. The proof of the main result is given in Section 5. The last two sections are devoted to various applications of the J-invariant and examples of motivic decompositions. In particular, we discuss the relations with canonical p-dimensions, degrees of zero-cycles, and the Rost invariant.

1. Chow motives and rational cycles

In the present section we follow the notation and definitions from [13, Ch. XII] (see also [26]).

Let X be a smooth projective irreducible variety over a field F. Let $\mathrm{CH}_i(X;\Lambda)$ be the Chow group of cycles of dimension i on X with coefficients in a commutative ring Λ. For simplicity we denote $\mathrm{CH}(X;\mathbb{Z})$ by $\mathrm{CH}(X)$.

1.1. Definition. – Following [13, § 63] an element $\phi\in\mathrm{CH}_{\dim X+d}(X\times Y;\Lambda)$ is called a *correspondence* between X and Y of degree d with coefficients in Λ. Let $\phi\in\mathrm{CH}(X\times Y;\Lambda)$ and $\psi\in\mathrm{CH}(Y\times Z;\Lambda)$ be correspondences of degrees d and e respectively. Then their product $\psi\circ\phi$ is defined by the formula $(\mathrm{pr}_{XZ})_*(\mathrm{pr}_{XY}^*(\phi)\cdot\mathrm{pr}_{YZ}^*(\psi))$ and has degree $d+e$. The *correspondence product* endows the group $\mathrm{CH}(X\times X;\Lambda)$ with a ring structure. The identity element of this ring is the class of the diagonal Δ_X. Given $\phi\in\mathrm{CH}(X\times X;\Lambda)$ of degree d we define a Λ-linear map $\mathrm{CH}_i(X;\Lambda)\to\mathrm{CH}_{i+d}(Y;\Lambda)$ by $\alpha\mapsto(\mathrm{pr}_Y)_*(\mathrm{pr}_X^*(\alpha)\cdot\phi)$. This map is called *realization* of ϕ and is denoted by $\phi_\star$. By definition $(\psi\circ\phi)_\star = \psi_\star\circ\phi_\star$. Given a correspondence ϕ we denote its *transpose* by ϕ^t.

1.2. Definition. – Following [13, § 64] let $\mathcal{M}(X;\Lambda)$ denote the *Chow motive* of X with Λ-coefficients and let $\mathcal{M}(X;\Lambda)(n) = \mathcal{M}(X;\Lambda)\otimes\Lambda(n)$ be the respective *twist* by the Tate motive. For simplicity we will denote $\mathcal{M}(X;\mathbb{Z})$ by $\mathcal{M}(X)$. Recall that morphisms between $\mathcal{M}(X;\Lambda)(n)$ and $\mathcal{M}(Y;\Lambda)(m)$ are given by correspondences of degree $n-m$. The group of endomorphisms $\mathrm{End}(\mathcal{M}(X;\Lambda))$ coincides with the Chow group $\mathrm{CH}_{\dim X}(X\times X;\Lambda)$. Observe that to provide a direct sum decomposition of $\mathcal{M}(X;\Lambda)$ is the same as to provide a family of pair-wise orthogonal idempotents $\phi_i\in\mathrm{End}(\mathcal{M}(X;\Lambda))$ such that $\sum_i\phi_i=\Delta_X$.

1.3. Definition. – Assume that a motive M is a direct sum of twisted Tate motives. In this case its Chow group $\mathrm{CH}(M)$ is a free abelian group. We define its *Poincaré polynomial* as

$$P(M,t)=\sum_{i\geq 0}a_i t^i,$$

where a_i is the rank of $\mathrm{CH}_i(M)$.

1.4. Definition. – Let L/F be a field extension. We say L is a *splitting field* of a smooth projective variety X or, equivalently, a variety X *splits* over L if the motive $\mathcal{M}(X;\mathbb{Z})$ is isomorphic over L to a finite direct sum of twisted Tate motives.

1.5. Example. – A variety X over a field F is called *cellular* if X has a proper descending filtration by closed subvarieties X_i such that each complement $X_i\setminus X_{i+1}$ is a disjoint union of affine spaces defined over F. According to [13, Corollary 66.4] if X is cellular, then X splits over F.

In particular, let G be a semisimple linear algebraic group over a field F and let X be a projective homogeneous G-variety. Assume that the group G splits over the generic point of X, i.e. $G_{F(X)}=G\times_F F(X)$ contains a split maximal torus defined over $F(X)$. Then $X_{F(X)}$ is a cellular variety and, therefore, $F(X)$ is a splitting field of X. Some concrete examples of such varieties are provided in 3.6.

1.6. Definition. – Assume X has a splitting field L. We will write $\mathrm{CH}(\overline{X};\Lambda)$ for $\mathrm{CH}(X_L;\Lambda)$ and $\overline{\mathrm{CH}}(X;\Lambda)$ for the image of the restriction map $\mathrm{CH}(X;\Lambda)\to\mathrm{CH}(\overline{X};\Lambda)$ (cf. [22, 1.2]). Similarly, we denote by $\mathcal{M}(\overline{X};\Lambda)$ the motive of X considered over L. If M is a direct summand of $\mathcal{M}(X;\Lambda)$, we denote by $\overline{M}$ the motive M_L. The elements of $\overline{\mathrm{CH}}(X;\Lambda)$ will be called *rational* cycles on X_L with respect to the field extension L/F and the coefficient ring Λ. If L' is another splitting field of X, then there is a chain of canonical isomorphisms $\mathrm{CH}(X_L)\simeq\mathrm{CH}(X_{LL'})\simeq\mathrm{CH}(X_{L'})$, where LL' is the composite of L and L'. Hence, the groups $\mathrm{CH}(\overline{X})$ and $\overline{\mathrm{CH}}(X)$ do not depend on the choice of L.

1.7. – According to [22, Remark 5.6] there is the Künneth decomposition $\mathrm{CH}(\overline{X}\times\overline{X})=\mathrm{CH}(\overline{X})\otimes\mathrm{CH}(\overline{X})$ and Poincaré duality holds for $\mathrm{CH}(\overline{X})$. The latter means that given a basis of $\mathrm{CH}(\overline{X})$ there is a dual one with respect to the non-degenerate pairing $(\alpha,\beta)\mapsto\deg(\alpha\cdot\beta)$, where deg is the degree map. In view of the Künneth decomposition the correspondence product of cycles in $\mathrm{CH}(\overline{X}\times\overline{X})$ is given by the formula