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QUANTITATIVE STOCHASTIC HOMOGENIZATION OF CONVEX INTEGRAL FUNCTIONALS

BY SCOTT N. ARMSTRONG AND CHARLES K. SMART

ABSTRACT. – We present quantitative results for the homogenization of uniformly convex integral functionals with random coefficients under independence assumptions. The main result is an error estimate for the Dirichlet problem which is algebraic (but sub-optimal) in the size of the error, but optimal in stochastic integrability. As an application, we obtain quenched $C^{0,1}$ estimates for local minimizers of such energy functionals.

RÉSUMÉ. – Nous présentons des résultats quantitatifs pour l'homogénéisation de fonctionnelles intégrales uniformément convexes avec coefficients aléatoires sous hypothèses d'indépendance. Le résultat principal est une estimation d'erreur pour le problème de Dirichlet qui est algébrique (mais sous-optimale) en la taille de l'erreur, mais optimale en intégrabilité stochastique. Comme application, nous obtenons des estimées $C^{0,1}$ pour les minimiseurs locaux de telles fonctionnelles d'énergie.

1. Introduction

1.1. Informal summary of results

We consider stochastic homogenization of the variational problem

$$\text{minimize} \quad \int_U L\left(Du(x), \frac{x}{\varepsilon}\right) dx \qquad \text{subject to} \qquad u \in g + H_0^1(U). \tag{1.1}$$

Here $0 < \varepsilon \ll 1$ is a small parameter, $U \subseteq \mathbb{R}^d$ is a smooth bounded domain and $g \in H^1(U)$ is given. The precise hypotheses on the Lagrangian L are given below; here we mention that $L(p,x)$ is uniformly convex in p and that L is a random field sampled by a given probability measure $\mathbb{P}$. The crucial hypothesis on the statistics of L is a *finite range of dependence* condition: roughly, for all Borel sets $U, V \subseteq \mathbb{R}^d$, the families $\{L(p,x) : p \in \mathbb{R}^d,\ x \in U\}$ and $\{L(p,x) : p \in \mathbb{R}^d,\ x \in V\}$ of random variables are assumed to be $\mathbb{P}$-independent provided that $\operatorname{dist}(U,V) \geq 1$.

An important special case of the model occurs if the Lagrangian is the quadratic form $L(p, x) = p \cdot A(x)p$. The corresponding Euler-Lagrange equation is then linear and the problem is equivalent to the stochastic homogenization of the equation

$$-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right) Du^{\varepsilon}\right) = 0. \tag{1.2}$$

This is also a continuum version of what is known in the probability literature as the *random conductance model*.

Dal Maso and Modica [8, 9] proved, in a somewhat more general setting, the basic *qualitative homogenization* result for (1.1): there exists a (deterministic) function $\overline{L} : \mathbb{R}^d \to \mathbb{R}$ called the *effective Lagrangian* such that, with probability one, the unique minimizer u^{ε} of (1.1) converges, as $\varepsilon \to 0$, to the unique minimizer of the variational problem

$$\text{minimize} \quad \int_U \overline{L}\left(Du(x)\right) \, dx \qquad \text{subject to} \qquad u \in g + H_0^1(U). \tag{1.3}$$

This result was a generalization to the nonlinear setting of earlier qualitative results for linear elliptic partial differential equations in divergence form due to Kozlov [18], Papanicolaou and Varadhan [26] and Yurinkskii [28], using new variational ideas based on subadditivity that were not present in earlier works.

An intense focus has recently emerged on building a quantitative theory of stochastic homogenization in the case of the linear Equation (1.2). This escalated significantly with the work of Gloria and Otto [15, 16], who proved optimal quantitative bounds for the energy density of modified correctors and then that of Gloria Neukamm and Otto [14, 13], who proved optimal bounds for the error in homogenization. These results were proved for discrete elliptic equations, but have been extended to the continuum setting in [17]. See also Mourrat [20, 21], Marahrens and Otto [19], Conlon and Spencer [7] as well as earlier works of Yurinskii [28], Naddaf and Spencer [24], Bourgeat and Piatnitski [6] and Boivin [5]. For some recent work on limit theorems for the stochastic fluctuations, see [23, 22, 25, 27, 4]. The analysis in the present paper was informed by some ideas from our previous work [1], which contained similar results for equations in nondivergence form.

In this paper, we present the first *quantitative* results for the homogenization of (1.1) which are also the first such results for divergence-form elliptic equations outside of the linear setting. We prove two main results: estimates for the L^2 and L^∞ error in homogenization of the Dirichlet problem, which is algebraic (yet sub-optimal) in its estimate of the typical size of the error, and essentially optimal in stochastic integrability; and a “stochastic higher regularity” result which states that local minimizers of (1.1), for a typical realization of the coefficients, satisfy the same *a priori* $C^{0,1}$ and $C^{1,\beta}$ regularity estimates as local minimizers of constant-coefficient energy functionals, down to microscopic and mesoscopic scales, respectively.

The first main result (Theorem 1.1) gives a sub-optimal algebraic error estimate in homogenization with strong stochastic integrability: it asserts roughly that, for any $s < d$, there exists an exponent $\alpha > 0$, depending on s, the dimension d and the constants controlling the uniform convexity of L and a constant $C \geq 1$, depending additionally on the given data, such that, for every $\delta \in (0,1]$,

$$\mathbb{P}\left[\exists \varepsilon \in (0,\delta],\ \fint_U |u^\varepsilon(x) - u_{\mathrm{hom}}(x)|^2\, dx \geq C\varepsilon^\alpha\right] \leq C \exp\left(-\delta^{-s}\right), \tag{1.4}$$

where u^ε and u_{hom} denote the unique minimizers in $g+H^1_0(U)$ of (1.1) and (1.3), respectively. Depending on the smoothness of the given Dirichlet boundary data g, this L^2 estimate may be upgraded to L^∞ by interpolating the latter between L^2 and $C^{0,\gamma}$ and using the nonlinear De Giorgi-Nash-Moser estimate. There is no loss in stochastic integrability in this interpolation and essentially no loss in the size of the error, since the exponent α is already sub-optimal. (See Corollary 4.2.) We remark that, at this stage in the development of the theory, we are less concerned with the sub-optimality of the size of the error than with the strength of the stochastic integrability; the former will be improved later. In (1.4) we have obtained the best possible stochastic integrability in the sense that such an estimate is false for $s > d$.

The second main result (Theorem 1.2) asserts that local minimizers of the energy functional in (1.1) are much smoother than minimizers for general functionals with measurable coefficients: it states roughly that any local minimizer u^ε of the energy functional satisfies the estimate

$$\sup_{x\in B_{1/2}\setminus B_\varepsilon} \frac{|u^\varepsilon(x) - u^\varepsilon(0)|}{|x|} \leq \mathcal{Y}\left(1 + \|u^\varepsilon\|_{L^2(B_1)}\right), \tag{1.5}$$

where $\mathcal{Y}$ is a random variable (i.e, it depends on the coefficients but not on u^ε) which, for any $s < d$, can be chosen to satisfy

$$\mathbb{E}\left[\exp(\mathcal{Y}^s)\right] < \infty.$$

This is a quenched Lipschitz estimate “down to the microscopic scale" since the left side of (1.5) is a finite difference approximation of $|Du^\varepsilon(0)|$.

The estimate (1.5) can be written in other forms, such as

$$\fint_{B_r} |Du^\varepsilon(x)|^2\, dx \leq C\left(1 + \|u^\varepsilon\|^2_{L^2(B_1)}\right) \quad \text{for every } r \in \left[\varepsilon\mathcal{Y}, \frac{1}{2}\right]. \tag{1.6}$$

The latter gives very good control of the spatial averages of the energy density of u^ε. As was shown by Gloria and Otto [15] in the linear setting, if the probability measure $\mathbb{P}$ satisfies a spectral gap hypothesis, then an estimate like (1.6) implies optimal bounds on the variance of the energy of, e.g., minimizers with periodic boundary conditions. In a future work, we will prove this and other optimal quantitative estimates from higher regularity estimates.

Theorem 1.2 also asserts that local minimizers behave even more smoothly on *meso*scopic scales (those of order ε^β for some $\beta \in (0,1)$) by giving an improvement of flatness estimate: see (1.16).

The proof of the error estimates, like the arguments of [8, 9], is variational and centers on the analysis of certain subadditive and superadditive energy quantities. However, the methods here differ substantially from those of [8, 9], as quantitative results present difficulties which do not appear in the qualitative theory and which require not just a harder analysis but also a new approach to the problem. The qualitative theory is based on the observation