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## SUBGROUP DYNAMICS AND $C^*$ -SIMPLICITY OF GROUPS OF HOMEOMORPHISMS

BY ADRIEN LE BOUDEC AND NICOLÁS MATTE BON

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**ABSTRACT.** – We study the uniformly recurrent subgroups of groups acting by homeomorphisms on a topological space. We prove a general result relating uniformly recurrent subgroups to rigid stabilizers of the action, and deduce a  $C^*$ -simplicity criterion based on the non-amenability of rigid stabilizers. As an application, we show that Thompson’s group  $V$  is  $C^*$ -simple, as well as groups of piecewise projective homeomorphisms of the real line. This provides examples of finitely presented  $C^*$ -simple groups without free subgroups. We prove that a branch group is either amenable or  $C^*$ -simple. We also prove the converse of a result of Haagerup and Olesen: if Thompson’s group  $F$  is non-amenable, then Thompson’s group  $T$  must be  $C^*$ -simple. Our results further provide sufficient conditions on a group of homeomorphisms under which uniformly recurrent subgroups can be completely classified. This applies to Thompson’s groups  $F$ ,  $T$  and  $V$ , for which we also deduce rigidity results for their minimal actions on compact spaces.

**RÉSUMÉ.** – Nous étudions les sous-groupes uniformément récurrents de groupes agissant par homéomorphismes sur un espace topologique. Nous prouvons un résultat général reliant les sous-groupes uniformément récurrents aux stabilisateurs rigides de l’action, et en déduisons un critère de  $C^*$ -simplicité basé sur la non moyennabilité des stabilisateurs rigides. Comme application, nous prouvons que le groupe de Thompson  $V$  est  $C^*$ -simple, de même que certains groupes d’homéomorphismes projectifs par morceaux de la droite réelle. Cela fournit des exemples de groupes finiment présentés qui sont  $C^*$ -simples et sans sous-groupes libres. Nous prouvons qu’un groupe branché est soit moyennable, soit  $C^*$ -simple. Nous prouvons également la réciproque d’un résultat de Haagerup et Olesen: si le groupe de Thompson  $F$  n’est pas moyennable alors le groupe de Thompson  $T$  est  $C^*$ -simple. Nos résultats fournissent de plus des conditions suffisantes sur un groupe d’homéomorphismes sous lesquelles les sous-groupes uniformément récurrents sont complètement compris. Cela s’applique aux groupes de Thompson, pour lesquels nous déduisons également des résultats de rigidité sur leurs actions sur des espaces compacts.

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## 1. Introduction

Let  $G$  be a second countable locally compact group. The set  $\text{Sub}(G)$  of all closed subgroups of  $G$  admits a natural topology, defined by Chabauty in [17]. This topology turns  $\text{Sub}(G)$  into a compact metrizable space, on which  $G$  acts continuously by conjugation.

The study of  $G$ -invariant Borel probability measures on  $\text{Sub}(G)$ , named *invariant random subgroups* (IRS's) after [2], is a fast-developing topic [2, 1, 27, 3]. In this paper we are interested in their topological counterparts, called *uniformly recurrent subgroups* (URSs) [24]. A uniformly recurrent subgroup is a closed, minimal,  $G$ -invariant subset  $\mathcal{H} \subset \text{Sub}(G)$ . We will denote by  $\text{URS}(G)$  the set of uniformly recurrent subgroups of  $G$ . Every normal subgroup  $N \trianglelefteq G$  gives rise to a URS of  $G$ , namely the singleton  $\{N\}$ . More interesting examples arise from minimal actions on compact spaces: if  $G$  acts minimally on a compact space  $X$ , then the closure of all point stabilizers in  $\text{Sub}(G)$  contains a unique URS, called the *stabilizer URS* of  $G \curvearrowright X$  [24].

When  $G$  has only countably many subgroups (e.g., if  $G$  is polycyclic), every IRS of  $G$  is atomic and every URS of  $G$  is finite, as follows from a standard Baire argument. Leaving aside this specific situation, there are important families of groups for which a precise description of the space  $\text{IRS}(G)$  has been obtained. Considerably less is known about URSs. For example if  $G$  is a lattice in a higher rank simple Lie group, the normal subgroup structure of  $G$  is described by Margulis' Normal Subgroups Theorem. While Stuck-Zimmer's Theorem [64] generalizes Margulis' NST to IRS's, it is an open question whether a similar result holds for URSs of higher rank lattices, even for the particular case of  $\text{SL}(3, \mathbb{Z})$  [24, Problem 5.4].

### 1.1. Micro-supported actions

In this paper we study the space  $\text{URS}(G)$  for countable groups  $G$  admitting a faithful action  $G \curvearrowright X$  on a topological space  $X$  such that for every non-empty open set  $U \subset X$ , the *rigid stabilizer*  $G_U$ , i.e., the pointwise stabilizer of  $X \setminus U$  in  $G$ , is non-trivial. Following [16], such an action will be called *micro-supported*. Note that this implies in particular that  $X$  has no isolated points.

The class of groups admitting a micro-supported action includes Thompson's groups  $F < T < V$  and many of their generalizations, groups of piecewise projective homeomorphisms of the real line [49], piecewise prescribed tree automorphism groups [41], branch groups (viewed as groups of homeomorphisms of the boundary of the rooted tree) [4], and topological full groups acting minimally on the Cantor set. These groups have uncountably many subgroups, and many examples in this class have a rich subgroup structure.

Our first result shows that many algebraic or analytic properties of rigid stabilizers are inherited by the uniformly recurrent subgroups of  $G$ . In the following theorem and everywhere in the paper, a uniformly recurrent subgroup  $\mathcal{H} \in \text{URS}(G)$  is said to have a group property if every  $H \in \mathcal{H}$  has the corresponding property.

**THEOREM 1.1** (see also Theorem 3.5). – *Let  $G$  be a countable group of homeomorphisms of a Hausdorff space  $X$ . Assume that for every non-empty open set  $U \subset X$ , the rigid stabilizer  $G_U$  is non-amenable (respectively contains free subgroups, is not elementary amenable, is*

not virtually solvable, is not locally finite). Then every non-trivial uniformly recurrent subgroup of  $G$  has the same property.

This result has applications to the study of  $C^*$ -simplicity; see Section 1.2.

We obtain stronger conclusions on the uniformly recurrent subgroups of  $G$  under additional assumptions on the action of  $G$  on  $X$ . Recall that when  $X$  is compact and  $G \curvearrowright X$  is minimal, the closure of all point stabilizers in  $\text{Sub}(G)$  contains a unique URS, called the *stabilizer URS* of  $G \curvearrowright X$ , and denoted  $\mathcal{S}_G(X)$  [24] (see Section 2 for details). The following result provides sufficient conditions under which  $\mathcal{S}_G(X)$  turns out to be the unique URS of  $G$ , apart from the points  $\{1\}$  and  $\{G\}$  (hereafter denoted  $1$  and  $G$ ). We say that  $G \curvearrowright X$  is an *extreme boundary action* if  $X$  is compact and the action is minimal and extremely proximal (see §2.1 for the definition of an extremely proximal action).

**THEOREM 1.2.** – *Let  $X$  be a compact Hausdorff space, and let  $G$  be a countable group of homeomorphisms of  $X$ . Assume that the following conditions are satisfied:*

- (i)  $G \curvearrowright X$  is an extreme boundary action;
- (ii) there is a basis for the topology consisting of open sets  $U \subset X$  such that the rigid stabilizer  $G_U$  admits no non-trivial finite or abelian quotients;
- (iii) the point stabilizers for the action  $G \curvearrowright X$  are maximal subgroups of  $G$ .

Then the only uniformly recurrent subgroups of  $G$  are  $1$ ,  $G$  and  $\mathcal{S}_G(X)$ .

We actually prove a more general result, see Corollary 3.12. Examples of groups to which this result applies are Thompson's groups  $T$  and  $V$ , as well as examples in the family of groups acting on trees  $G(F, F')$  (see Section 1.3 and Section 1.5). In particular, this provides examples of finitely generated groups  $G$  (with uncountably many subgroups) for which the space  $\text{URS}(G)$  is completely understood. In the case of Thompson's groups, we deduce from this lack of URSs rigidity results about their minimal actions on compact spaces (see §1.3).

## 1.2. Application to $C^*$ -simplicity

A group  $G$  is said to be  $C^*$ -simple if its reduced  $C^*$ -algebra  $C_{\text{red}}^*(G)$  is simple. This property naturally arises in the study of unitary representations:  $G$  is  $C^*$ -simple if and only if every unitary representation of  $G$  that is weakly contained in the left-regular representation  $\lambda_G$  is actually weakly equivalent to  $\lambda_G$  [33]. Since amenability of a group  $G$  is characterized by the fact that the trivial representation of  $G$  is weakly contained in  $\lambda_G$ , a non-trivial amenable group is never  $C^*$ -simple.

The first historical  $C^*$ -simplicity result was Powers' proof that the reduced  $C^*$ -algebra of the free group  $\mathbb{F}_2$  is simple [60]. The methods employed by Powers have then been generalized in several different ways, and various classes of groups have been shown to be  $C^*$ -simple. We refer to [33, Proposition 11] (see also the references given there), and to Corollary 12 therein for a list of important examples of groups to which these methods have been applied.

Problems related to  $C^*$ -simplicity recently received new attention [38, 8, 61, 41, 39, 31]. A characterization of  $C^*$ -simplicity in terms of boundary actions was obtained by Kalantar and Kennedy: a countable group  $G$  is  $C^*$ -simple if and only if  $G$  acts topologically freely on its Furstenberg boundary; equivalently,  $G$  admits some topologically free boundary action [38] (we recall the terminology in §2.1). By developing a systematic approach based on