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ON THE SUBCONVEXITY PROBLEM FOR L-FUNCTIONS ON GL(3)

BY VALENTIN BLOMER AND JACK BUTTCANE

ABSTRACT. – Let f be a cusp form for the group $\mathrm{SL}(3,\mathbb{Z})$ with Langlands parameter μ and associated L-function $L(s,f)$. If μ is in generic position, i.e., away from the Weyl chamber walls and away from self-dual forms, we prove the subconvexity bound $L(1/2,f) \ll \|\mu\|^{\frac{3}{4}-\frac{1}{120000}}$.

RÉSUMÉ. – Soit f une forme parabolique pour le groupe $\mathrm{SL}(3,\mathbb{Z})$. Désignons par μ ses paramètres de Langlands et $L(s,f)$ sa fonction L. Si μ est en position générique, dans le sens qu'il est à la fois loin des murs des chambres de Weyl et loin des formes auto duales, on démontre la borne sous-convexe $L(1/2,f) \ll \|\mu\|^{\frac{3}{4}-\frac{1}{120000}}$.

1. Introduction

1.1. The main result

Analytic number theory on higher rank groups has recently seen substantial advances. One of the most challenging touchstones for the strength of available techniques is the subconvexity problem for automorphic L-functions. We recall that subconvexity refers to an estimate of an automorphic L-function on the critical line that is superior, usually with a power saving, to the generic convexity bound in one or more of the defining parameters of the underlying automorphic form. This is a step towards the Lindelöf hypothesis – currently unknown for any L-function, even the Riemann zeta function – and usually requires the most advanced machinery (as compared, for instance, to non-vanishing questions that can often be achieved by softer methods). Subconvexity has been achieved for L-functions on GL(2) in full generality over arbitrary number fields [22]. In higher rank, the available results become very sporadic.

For a fixed, self-dual Maaß form for $\mathrm{SL}_3(\mathbb{Z})$, the first breakthrough was achieved by Xiaoqing Li [21] who solved the subconvexity problem in the t-aspect. This was generalized

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by Munshi [24] to arbitrary fixed Maaß forms. Similar results are available for twists by Dirichlet characters [1, 26, 25]. All of these results fall into the category of GL(1) twists of a fixed Maaß form and use mainly GL(1) and GL(2) tools (enhanced by the GL(3) Voronoi formula). Subconvexity in terms of genuine parameters of a GL(3) automorphic L-function (level or spectral parameter) has resisted all attempts so far and seems to require a completely new set of methods. This is what we achieve in this paper.

For the first time, we will go beyond GL(1) twists and prove a prototype of a genuine GL(3) subconvexity result using the full power of the spectral theory of automorphic forms on GL(3). The main tool is a higher rank (relative) trace formula, and the key novelty in this paper is that the off-diagonal terms on the geometric side are not just estimated, but subject to further transformations and averaged in a non-trivial way, thereby detecting arithmetic oscillation almost on the level of (best possible) square root cancelation. This is the first such instance in higher rank, and we hope that these ideas will open the door to more advanced applications of trace formulae in higher rank in the future (perhaps also in the context of the Selberg trace formula). Applications of these methods and ideas include, for instance, non-vanishing results for L-functions on GL(3) and GSp(4) [27, 3, 16], subonvexity results for non-spherical L-functions [5], best-possible large sieve inequalities on GL(3) [29, 4] and equidistibution results of Satake parameters [12].

We postpone a more precise discussion of the new methodological novelties for a moment and turn to the description of our main result. For an automorphic representation π we denote its Langlands parameter by $\mu = \mu_\pi = (\mu_1, \mu_2, \mu_3)$. This is a triple of complex numbers satisfying $\mu_1 + \mu_2 + \mu_3 = 0$, normalized such that the (archimedean) Ramanujan conjecture predicts $\mu \in (i\mathbb{R})^3$. Let π_0 be an everywhere unramified automorphic representation with Langlands parameter $\mu_0 = (\mu_{0,1}, \mu_{0,2}, \mu_{0,3})$. We assume that μ_0 is in *generic* position, i.e., there exist constants $C > c > 0$ such that

$$c \leqslant \frac{|\mu_{0,j}|}{\|\mu\|} \leqslant C \quad (1 \leqslant j \leqslant 3), \quad \text{and} \quad c \leqslant \frac{|\mu_{0,i} - \mu_{0,j}|}{\|\mu\|} \leqslant C \quad (1 \leqslant i < j \leqslant 3). \tag{1.1}$$

This set describes two cones in each Weyl chamber away from the walls and away from the self-dual forms, and covers 99% of all Maaß forms (choosing c and C appropriately) by Weyl's law [19, Theorem 1.1][(1)]. For the rest of the paper we fix c and C, and all implied constants may depend on them. The convexity bound for L-functions associated with such representations states $L(s, \pi_0) \ll \|\mu_0\|^{3/4+\varepsilon}$.

Theorem 1. – *Let* $\pi_0 \subseteq L^2(\mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R}))$ *be an irreducible, cuspidal, spherical representation with large Langlands parameter* μ_0 *in generic position. Then*

$$L(1/2, \pi_0) \ll \|\mu_0\|^{\frac{3}{4} - \frac{1}{120000}}.$$

We remark that the same proof works almost literally for any fixed point on the critical line and produces $L(1/2 + it, \pi_0) \ll_t \|\mu_0\|^{\frac{3}{4} - \frac{1}{120000}}$ with polynomial dependence in t. It also works for Maaß forms for fixed congruence subgroups $\Gamma_0(N) \subseteq \mathrm{SL}_3(\mathbb{Z})$, again with polynomial dependence on N. In an effort to keep the length of the paper at reasonable size, we have not tried to obtain the best exponent available by our method.

(1) Which assumes level $N \geqslant 3$ in order to avoid elliptic elements in the subgroup, but the proof can be modified to treat the level 1 case as well.

As mentioned before, the main tool is the GL(3) Kuznetsov formula that was successively refined, most notably in [11], and has recently been used for a variety of applications. The starting point is an amplified fourth moment, averaged over representations with Langlands parameter in a small ball about μ_0. We insert an approximate functional equation and apply Poisson summation in all four variables. It is instructive to compare this with the GL(2) version, which was worked out by Iwaniec [17, Theorem 4] more than 20 years ago. While for GL(2) a *second* moment suffices, in rank 2 a fourth moment is necessary, and the method requires an extremely delicate analysis of Kloosterman sums and special functions. There are several other new phenomena in higher rank that will be discussed in due course. On the technical side, we need very precise estimates for the four-fold Fourier transform of the kernel function of the Kuznetsov transform associated to the long Weyl element. Ultimately this amounts to the analysis of a multi-dimensional oscillatory integral with degenerate and non-degenerate stationary points to which we apply, among other things, Morse theory in the form of a theorem of Milnor and Thom. Several auxiliary results on special functions and integral transforms associated with the group $\mathrm{GL}_3(\mathbb{R})$ may be useful in other situations.

The excluded situations in Theorem 1, i.e., forms close to self-dual forms and close to the walls of the Weyl chambers, are exceptional for two different reasons: for self-dual forms the conductor of the L-function drops so that instead of a fourth moment a sixth moment would be necessary (Theorem 1 remains true in the self-dual case, too, but is worse than the convexity bound). Close to the Weyl chambers, on the other hand, the spectral measure drops, so that the spectral average becomes less powerful. Notice that possible exceptional spectral parameters (i.e., violating the archimedean Ramanujan conjecture) lie on the Weyl chamber walls, so that these are in particular excluded; this simplifies some of the forthcoming arguments, but is not essential to the method.

1.2. A heuristic roadmap

It might be useful to give a short informal description of the proof which reflects reality – if at all – only in a very vague sense, but may guide the reader through the argument. The mean value

$$\sum_{\mu=\mu_0+O(1)} |L(1/2,\pi)|^4 \tag{1.2}$$

contains about T^3 terms, where $T = \|\mu_0\|$. If we can show that the off-diagonal term is $\ll T^{3-\delta}$ for some $\delta > 0$, then the amplification method will prove subconvexity. Our amplifier has length $L = T^{\lambda}$ for some very small $\lambda > 0$, but for simplicity we suppress the amplifier in the present discussion. By an approximate functional equation we have

$$|L(1/2,\pi)|^4 \approx T^{-3} \sum_{m_1,m_2,n_1,n_2 \asymp T^{3/2}} A_\pi(m_2,n_1)\overline{A_\pi(m_1,n_2)},$$

where here and throughout the section we do not display smooth weight functions and A_π are the Fourier-Whittaker coefficients of π. The contribution of the long Weyl element of the Kuznetsov formula is roughly of the shape

$$T^{-3} \sum_{m_1,m_2,n_1,n_2 \asymp T^{3/2}} \sum_{D_1,D_2} \frac{S(n_1,m_2,m_1,n_2;D_1,D_2)}{D_1D_2} \Phi\left(\frac{n_1m_1D_2}{D_1^2}, \frac{n_2m_2D_1}{D_2^2}\right), \tag{1.3}$$