ISSN 0012-9593

quatrième série - tome 53 fascicule 3 mai-juin 2020

ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE

Radu IGNAT & Luc NGUYEN & Valeriy SLASTIKOV & Arghir ZARNESCU

On the uniqueness of minimisers of Ginzburg-Landau functionals

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 53, 2020, p. 589 à 613

ON THE UNIQUENESS OF MINIMISERS OF GINZBURG-LANDAU FUNCTIONALS

BY RADU IGNAT, LUC NGUYEN, VALERIY SLASTIKOV AND ARGHIR ZARNESCU

ABSTRACT. – We provide necessary and sufficient conditions for the uniqueness of minimisers of the Ginzburg-Landau functional for $\mathbb{R}^n$-valued maps under a suitable convexity assumption on the potential and for $H^{1/2} \cap L^\infty$ boundary data that is non-negative in a fixed direction $e \in \mathbb{S}^{n-1}$. Furthermore, we show that, when minimisers are not unique, the set of minimisers is generated from any of its elements using appropriate orthogonal transformations of $\mathbb{R}^n$. We also prove corresponding results for harmonic maps with values into $\mathbb{S}^{n-1}$.

RÉSUMÉ. – Nous montrons des conditions nécessaires et suffisantes pour l'unicité des minimiseurs de la fonctionnelle de Ginzburg-Landau sous une hypothèse de convexité du potentiel et pour des données au bord dans $H^{1/2} \cap L^\infty$ qui sont positives dans une direction fixée. De plus, nous prouvons que si le minimiseur n'est pas unique, alors l'ensemble des minimiseurs est généré par une certaine classe de transformations orthogonales appliquées à un minimiseur quelconque. Nous montrons aussi des résultats similaires pour les applications harmoniques à valeurs dans la sphère unité.

1. Introduction and main results

We consider the following Ginzburg-Landau type energy functional

$$E_\varepsilon(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1-|u|^2)\right] dx, \tag{1.1}$$

where $\varepsilon > 0$, $\Omega \subset \mathbb{R}^m$ ($m \geq 1$) is a bounded domain (i.e., open connected set) with smooth boundary $\partial\Omega$ and the potential $W \in C^1((-\infty, 1]; \mathbb{R})$ satisfies

$$W(0) = 0,\ W(t) > 0 \text{ for all } t \in (-\infty, 1] \setminus \{0\},\ W \text{ is strictly convex.} \tag{1.2}$$

We investigate the minimisers of the energy E_ε over the set

$$\mathscr{A} := \{u \in H^1(\Omega; \mathbb{R}^n) : u = u_{\rm bd} \text{ on } \partial\Omega\}, \quad n \geq 1, \tag{1.3}$$

consisting of H^1 maps with a given boundary data (in the sense of $H^{1/2}$-trace on $\partial\Omega$):

$$u_{\rm bd} \in H^{1/2} \cap L^\infty(\partial\Omega; \mathbb{R}^n).$$

doi:10.24033/asens.2429

In particular, we are interested in the question of uniqueness (or its failure) for the minimisers of E_ε in $\mathscr{A}$ for all range of $\varepsilon > 0$.

In the case $\varepsilon > 0$ is large enough and (1.2) holds, the energy E_ε is strictly convex over the convex set $\mathscr{A}$ (see Remark 3.3); as E_ε is lower semicontinuous in the weak H^1-topology and coercive over $\mathscr{A}$, there is a unique critical point u_ε of E_ε over $\mathscr{A}$ and this critical point is the global minimiser of the energy E_ε. The case when $\varepsilon > 0$ is "not too large" is much more complex and, in general, one has to impose additional assumptions on the potential W or/and the boundary data to understand the uniqueness of minimisers and its failure even in the limit $\varepsilon \to 0$ (i.e., the related case of minimizing harmonic maps with values into $\mathbb{S}^{n-1}$).

There exists a large literature using various methods that addresses the question of uniqueness of minimisers in the framework of Ginzburg-Landau type models and the related harmonic map problem. See e.g., Bethuel, Brezis and Hélein [2], Mironescu [22], Pacard and Rivière [23], Ye and Zhou [28], Farina and Mironescu [7], Millot and Pisante [21], Pisante [24], Gustafson [9], Jäger and Kaul [17, 18], Sandier and Shafrir [25, 26] and the references therein.

The approach of this work is based on the method that we previously successfully used in the investigation of stability and minimality properties of the critical points in the context of Landau-de Gennes model of nematic liquid crystals [5, 12, 13, 14]. We refer to this tool as the Hardy decomposition technique (see Lemma A.1. in [12]). We show that under the global assumption (1.2) on the potential W (namely its convexity), and an additional assumption on the boundary data (namely its non-negativity in a fixed direction), we can use Hardy decomposition method to provide simple yet quite general proofs of uniqueness of minimisers of energy E_ε and to characterize its failure. In our forthcoming paper [16] we will provide sufficient conditions for uniqueness of minimisers under less restrictive conditions on the potential, but imposing more restrictive conditions on the boundary data.

We begin the exposition with providing a simple result on the uniqueness and symmetry of minimisers, which will be subsequently extended in a much more general setting. Assume $\Omega \subset \mathbb{R}^2$ is the unit disk, $u : \Omega \to \mathbb{R}^3$, and the boundary data (1) carries a given winding number $k \in \mathbb{Z} \setminus \{0\}$ on $\partial\Omega$, namely

$$u_{\mathrm{bd}}(\cos\varphi, \sin\varphi) = (\cos(k\varphi), \sin(k\varphi), 0) \in \mathbb{S}^1 \times \{0\} \subset \mathbb{R}^3, \quad \forall \varphi \in [0, 2\pi). \tag{1.4}$$

By restricting E_ε to a subset of $\mathscr{A}$ (with $n = 3$) consisting of suitable rotationally symmetric maps (see condition (4.1) below), it is not hard to see that E_ε admits critical points of the form

$$u_\varepsilon(r\cos\varphi, r\sin\varphi) := f_\varepsilon(r)(\cos(k\varphi), \sin(k\varphi), 0) \pm g_\varepsilon(r)(0, 0, 1),\ r \in (0, 1),\ \varphi \in [0, 2\pi), \tag{1.5}$$

where the couple $(f_\varepsilon, g_\varepsilon)$ of radial profiles solves the system

$$\begin{cases} -f_\varepsilon'' - \frac{1}{r} f_\varepsilon' + \frac{k^2}{r^2} f_\varepsilon &= \frac{1}{\varepsilon^2} f_\varepsilon W'(1 - f_\varepsilon^2 - g_\varepsilon^2) \\ -g_\varepsilon'' - \frac{1}{r} g_\varepsilon' &= \frac{1}{\varepsilon^2} g_\varepsilon W'(1 - f_\varepsilon^2 - g_\varepsilon^2) \end{cases} \quad \text{in } (0, 1), \tag{1.6}$$

subject to the boundary conditions

$$f_\varepsilon(0) = 0,\ f_\varepsilon(1) = 1,\ g_\varepsilon'(0) = 0,\ g_\varepsilon(1) = 0. \tag{1.7}$$

(1) We note that the map u_{bd} in (1.4) is non-negative in e_3-direction.

Our first result is the following:

THEOREM 1.1. – *Let* $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ *be the unit disk in* $\mathbb{R}^2$, $u : \Omega \to \mathbb{R}^3$, $W \in C^1((-\infty, 1]; \mathbb{R})$ *satisfy* (1.2), *and* u_{bd} *be the boundary data given by* (1.4) *where* $k \in \mathbb{Z} \setminus \{0\}$ *is a given integer. Then the following conclusions hold.*

1. *For every* $\varepsilon > 0$, *any minimiser* u_ε *of the energy* E_ε *in the set* $\mathscr{A}$ *(with* $n = 3$*) has the representation* (1.5) *where* $f_\varepsilon > 0$, $g_\varepsilon \geq 0$ *in* $(0, 1)$ *and* $(f_\varepsilon, g_\varepsilon)$ *solves* (1.6)-(1.7).
2. *There exists* $\varepsilon_k > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_k)$, (1.6)-(1.7) *has a unique solution* $(f_\varepsilon, g_\varepsilon)$ *with* $g_\varepsilon > 0$, *and for* $\varepsilon \in [\varepsilon_k, \infty)$, *no such solution exists.*
3. *For every* $\varepsilon > 0$, *there is exactly one solution* $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)$ *of* (1.6)-(1.7) *satisfying* $\tilde{f}_\varepsilon > 0$, $\tilde{g}_\varepsilon \equiv 0$.
4. *For every* $\varepsilon \in (0, \varepsilon_k)$, E_ε *has exactly two minimisers that are given by* $(f_\varepsilon, \pm g_\varepsilon)$ *via* (1.5) *with* $g_\varepsilon > 0$, *while the critical point of* E_ε *corresponding to* $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon \equiv 0)$ *via* (1.5) *is unstable. For every* $\varepsilon \in [\varepsilon_k, \infty)$, E_ε *has a unique minimiser which corresponds to* $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon \equiv 0)$.

REMARK 1.2. – *We will prove in Proposition 4.2 that the critical values* ε_k *in Theorem 1.1 satisfy* $\varepsilon_k = \varepsilon_{-k}$ *for every* $k \geq 1$ *and the sequence* $(\varepsilon_k)_{k\geq 1}$ *is increasing. Moreover, for every* $k \in \mathbb{Z} \setminus \{0\}$, *the two minimisers of* E_ε *over* $\mathscr{A}$ *given by the solution* $(f_\varepsilon, \pm g_\varepsilon)$ *of the system* (1.6)-(1.7) *with* $g_\varepsilon > 0$ *converge strongly in* $H^2(\Omega)$ *as* $\varepsilon \to 0$ *to the two minimizing harmonic maps (see e.g.,* [1]*):*

$$(r, \varphi) \in (0, 1) \times [0, 2\pi) \mapsto \left(\frac{2r^k}{1 + r^{2k}} \cos(k\varphi), \frac{2r^k}{1 + r^{2k}} \sin(k\varphi), \pm \frac{1 - r^{2k}}{1 + r^{2k}} \right) \in \mathbb{S}^2.$$

The situation presented above indicates that uniqueness of minimisers can be lost in a discrete way (two alternative minimisers). This is due to an additional degree of freedom in the target space and becomes evident in a more general setting, when we work with $\mathbb{R}^n$-valued maps: if an isometry of $\mathbb{R}^n$ does not change the boundary data, then it transforms a minimiser into a minimiser. This is an important feature of our results that will be presented in the general setting for arbitrary dimensions $m \geq 1$ and $n \geq 1$ of base and target spaces, respectively. The main restriction in our treatment is the assumption that the boundary data is non-negative in a (fixed) direction $e \in \mathbb{S}^{n-1}$, i.e.,

(1.8) $$u_{\text{bd}} \cdot e \geq 0 \quad \mathscr{H}^{m-1}\text{-a.e. in } \partial\Omega.$$

Before formulating our main results we need to provide some basic definitions. In the rest of the paper we adopt the following notation: for any integrable $\mathbb{R}^n$-valued map u on a measurable set ω we denote the essential image of u on ω by

(1.9) $$u(\omega) = \{u(x) : x \in \omega \text{ is a Lebesgue point of } u\}.$$

It is clear that for a continuous function u the above definition coincides with the (standard) range of u on ω. We also define $\operatorname{Span} u(\omega)$ to be the smallest vector subspace of $\mathbb{R}^n$ containing $u(\omega)$.

The following result establishes the minimizing property and uniqueness (up to certain isometries) for *critical points* of E_ε that are *positive* in a fixed direction $e \in \mathbb{S}^{n-1}$ inside the domain $\Omega \subset \mathbb{R}^m$.