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Christopher HENDERSON, Stanley SNELSON
& Andrei TARFULEA

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Annales Scientifiques de l'École Normale Supérieure,
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Tél. : (33) 1 44 32 20 88.
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CLASSICAL SOLUTIONS OF THE BOLTZMANN EQUATION WITH IRREGULAR INITIAL DATA

BY CHRISTOPHER HENDERSON, STANLEY SNELSON AND ANDREI TARFULEA

ABSTRACT. – This article considers the spatially inhomogeneous, non-cutoff Boltzmann equation. We construct a large-data classical solution given bounded, measurable initial data with uniform polynomial decay of mild order in the velocity variable. Our result requires no assumption of strict positivity for the initial data, except locally in some small ball in phase space. We also obtain existence results for weak solutions when our decay and positivity assumptions for the initial data are relaxed.

Because the regularity of our solutions may degenerate as t tends to 0, uniqueness is a challenging issue. We establish weak-strong uniqueness under the additional assumption that the initial data possesses no vacuum regions and is Hölder continuous.

As an application of our short-time existence theorem, we prove global existence near equilibrium for bounded, measurable initial data that decays at a finite polynomial rate in velocity.

RÉSUMÉ. – Cet article étudie l'équation de Boltzmann inhomogène en espace sans troncature angulaire. En supposant la donnée initiale mesurable et bornée à décroissance polynomiale d'ordre limité en la variable de vitesse, on construit une solution classique. Aucune hypothèse de positivité stricte de la donnée initiale n'est nécessaire, mais le résultat repose sur une hypothèse locale de positivité stricte sur une petite boule dans l'espace des phases. On obtient l'existence de solutions faibles en relâchant les hypothèses de décroissance et de positivité.

La question d'unicité est rendue difficile car la régularité des solutions peut dégénérer quand t tend vers 0. On établit l'unicité faible-forte sous l'hypothèse supplémentaire pour la donnée initiale: absence de région vide et continuité de Hölder.

En application du résultat d'existence en temps court, on prouve l'existence globale près d'un équilibre pour une donnée initiale mesurable qui décroit polynomialement en la vitesse.

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1. Introduction

We consider the Boltzmann equation, a fundamental kinetic integro-differential equation from statistical physics [18, 66, 19, 21, 70]. The unknown function $f(t,x,v) \geq 0$ models the particle density of a diffuse gas in phase space at time $t \geq 0$, location $x \in \mathbb{R}^3$, and velocity $v \in \mathbb{R}^3$. The equation reads

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f), \tag{1.1}$$

where the left-hand side is a transport term, and $Q(f, g)$ is the Boltzmann collision operator with *non-cutoff* collision kernel, which we describe in detail below.

The purpose of this article is to develop a well-posedness theory for (1.1) on a time interval $[0, T]$, making minimal assumptions on the initial data $f_{\text{in}}(x,v) \geq 0$. In particular, we would like our local existence theory to properly encapsulate the *regularizing effect* of the non-cutoff Boltzmann equation. This effect comes from the nonlocal diffusion produced by $Q(f, f)$ in the velocity variable, and has been studied extensively, as we survey below in Section 1.4.2. In light of this regularizing effect, it is natural and desirable to construct a solution f with initial data in a low-regularity (ideally zeroth-order) space, such that f has at least enough regularity for positive times to evaluate the equation in a pointwise sense. However, so far this has only been achieved in the close-to-equilibrium [13, 65] and space homogeneous (i.e x-independent) [32] regimes. For the general case, essentially all of the local existence results for classical solutions in the literature [7, 8, 10, 61, 40, 42] require f_{in} to lie in a weighted Sobolev space of order at least 4. The current article fills this gap by constructing a solution with initial data in a weighted L^∞-space.

Another goal of our analysis is to optimize the requirement on the decay of f_{in} for large velocities. Because of the nonlocality of Q, decay of solutions is intimately tied to regularity, and since we work in the physical regime $\gamma \leq 0$ (see (1.3)), the decay of f for positive times is limited by the decay of f_{in}. In our main existence result, we require f_{in} to have pointwise polynomial decay of order $3 + 2s$, where $2s \in (0, 2)$ is the order of the diffusion (see (1.4)). In particular, the energy density $\int_{\mathbb{R}^3} |v|^2 f(t,x,v)\, \mathrm{d}v$ of our solutions may be infinite, which places them outside the regime where the conditional regularity estimates of Imbert-Silvestre [48] may be applied out of the box.

The possible presence of vacuum regions in the initial data is a key source of difficulty. The regularization coming from $Q(f, f)$ relies on positivity properties of f, in a complex way that reflects the nonlocality of Q. In the space homogeneous setting, conservation of mass provides sufficient positivity of f for free. The close-to-equilibrium assumption would also ensure that f has regions of strict positivity at all times. By contrast, in the case of general initial data, any lower bounds for f must degenerate at a severe rate as $t \searrow 0$, which impacts the regularity of the solution for small times and causes complications for the well-posedness theory. Our main existence theorem requires a weak positivity assumption, namely that f_{in} is uniformly positive in some small ball in phase space.

We consider solutions posed on the spatial domain $\mathbb{R}^3$, with no assumption that the solution or the initial data decay for large $|x|$. This regime includes the physically important example of a localized disturbance away from a Maxwellian equilibrium $M(v) = c_1 e^{-c_2|v|^2}$; that is $f = M(v) + g(t,x,v)$, where $g(t,x,v) \to 0$ as $|x| \to \infty$ but g is not necessarily small. Our regime also includes spatially periodic solutions as a special case. The lack of integrability in x is a nontrivial source of difficulty and in particular makes energy methods much less convenient. Also, the total mass, energy, and entropy of the solution could be infinite, so we do not have access to the usual bounds coming from conservation of mass and energy and monotonicity of entropy.

In Section 1.4.1, we give a more complete bibliography of well-posedness results for (1.1).

1.1. The collision operator

Boltzmann's collision operator is a bilinear integro-differential operator defined for functions $f, g : \mathbb{R}^3 \to \mathbb{R}$ by

$$Q(f,g) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)[f(v'_*)g(v') - f(v_*)g(v)] \, \mathrm{d}\sigma \, \mathrm{d}v_*. \tag{1.2}$$

Because collisions are assumed to be elastic, the pre- and post-collisional velocities all lie on a sphere of diameter $|v - v_*|$ parameterized by $\sigma \in \mathbb{S}^2$, and are related by the formulas

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

We take the standard *non-cutoff* collision kernel B defined by

$$B(v - v_*, \sigma) = b(\cos\theta)|v - v_*|^\gamma, \quad \cos\theta = \sigma \cdot \frac{v - v_*}{|v - v_*|}, \tag{1.3}$$

for some $\gamma > -3$. The angular cross-section b is singular as θ (the angle between pre- and post-collisional velocities) approaches 0 and satisfies the bounds

$$c_b \theta^{-1-2s} \leq b(\cos\theta)\sin\theta \leq \frac{1}{c_b}\theta^{-1-2s}, \tag{1.4}$$

for some $c_b > 0$ and $s \in (0, 1)$.

This implies b has the asymptotics $b(\cos\theta) \approx \theta^{-2-2s}$ as $\theta \to 0$. The parameters γ and s reflect the modeling choices made in defining $Q(f, g)$. When electrostatic interactions between particles are governed by an inverse-square-law potential of the form $\phi(x) = c|x|^{1-p}$ for some $p > 2$, then one has $\gamma = (p-5)/(p-1)$ and $s = 1/(p-1)$. As is common in the literature, we consider arbitrary pairs (γ, s) and disregard the parameter p.

For our main results, we assume

$$\gamma < 0,$$

but otherwise, we do not place any restriction on γ and s. The integral in (1.2) has two singularities: as $\theta \to 0$, and as $v_* \to v$. The non-integrable singularity at $\theta \approx 0$ (grazing collisions), which is related to the long-range interactions taken into account by the physical model, is the source of the regularizing properties of the operator Q.

1.2. Main results

For $1 \leq p \leq \infty$, define the velocity-weighted L^p norms

$$\|f\|_{L^p_q(\mathbb{R}^3)} = \|\langle v\rangle^q f\|_{L^p(\mathbb{R}^3)}, \quad \|f\|_{L^p_q(\Omega\times\mathbb{R}^3)} = \|\langle v\rangle^q f\|_{L^p(\Omega\times\mathbb{R}^3_v)}$$

where Ω is any subset of $\mathbb{R}^3_x$ or $[0, \infty) \times \mathbb{R}^3_x$.

Our results involve kinetic Hölder spaces C^β_ℓ that are defined precisely in Section 2.1 below. These spaces are based on a distance d_ℓ that is adapted to the scaling and translation