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ISOSPECTRAL CONNECTIONS, ERGODICITY OF FRAME FLOWS, AND POLYNOMIAL MAPS BETWEEN SPHERES

BY MIHAJLO CEKIĆ AND THIBAULT LEFEUVRE

Dedicated to the memory of Steve Zelditch

ABSTRACT. – We show that on closed negatively curved Riemannian manifolds with simple length spectrum, the spectrum of the Bochner Laplacian determines both the isomorphism class of the vector bundle and the connection up to gauge under a low-rank assumption. We also show that flows of frames on low-rank frame bundles extending the geodesic flow in negative curvature are ergodic whenever the bundle admits no holonomy reduction. This is achieved by exhibiting a link between these problems and the classification of polynomial maps between spheres in real algebraic geometry.

RÉSUMÉ. – Nous montrons que, sur les variétés riemanniennes fermées à courbure strictement négative et à spectre de longueur simple, le spectre du laplacien de Bochner détermine à la fois la classe d'isomorphisme du fibré vectoriel et la connexion à jauge près sous une hypothèse de rang faible. De plus, nous montrons que les flots partiellement hyperboliques obtenus comme extensions du flot géodésique à certains fibrés des repères de faible rang (sur des variétés à courbure strictement négative) sont ergodiques dès lors que le fibré n'admet aucune réduction d'holonomie. Ces résultats sont obtenus en établissant un lien entre ces problèmes et la classification des applications polynomiales entre sphères en géométrie algébrique réelle.

1. Introduction

The purpose of this article is to exhibit a link between the following three a priori unrelated problems:

- In real algebraic geometry: the (non-)existence of polynomial mappings between spheres.
- In dynamical systems: the ergodicity of certain partially hyperbolic dynamical systems obtained as isometric extensions of the geodesic flow over negatively curved Riemannian manifolds.

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— In spectral theory: Kac's isospectral problem "*Can one hear the shape of a drum*?" for the Bochner (or connection) Laplacian.

1.1. Background

In order to state the main theorems, we introduce some notation.

1.1.1. *Polynomial maps between spheres*. – Our main results will be formulated under a certain *low-rank* assumption, which is governed by the quantity $q(n) \in \mathbb{Z}_{\geq 1}$ from real algebraic geometry. Namely, for $n \in \mathbb{Z}_{\geq 1}$, define $q(n)$ to be the *least integer* $r \in \mathbb{Z}_{\geq 1}$ for which there exists a *non-constant polynomial map*

$$\mathbb{S}^n \to \mathbb{S}^r,$$

where $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the unit sphere; by polynomial map we mean the restriction to $\mathbb{S}^n$ of a polynomial map $\mathbb{R}^{n+1} \to \mathbb{R}^{r+1}$. It is known that $q(n)$ is even for $n \geq 2$, and a result due to Wood [47] implies

$$n/2 < q(n) \leq n, \qquad q(2^k) = 2^k, \quad k \in \mathbb{Z}_{\geq 0}.$$

The first few values are given by:

$$q(2) = q(3) = 2, \quad q(4) = \cdots = q(7) = 4, \quad q(8) = \cdots = q(15) = 8.$$

For more properties of $q(n)$, see §2 below.

1.1.2. *Connections*. – Let M be a smooth closed connected manifold. Let $\mathbf{A}^{\mathbb{R}}$ (resp. $\mathbf{A}^{\mathbb{C}}$) be the *moduli space* of orthogonal connections on oriented Euclidean vector bundles (resp. unitary connections on Hermitian vector bundles) over M. Write $\mathbf{A}^{\mathbb{F}}_r \subset \mathbf{A}^{\mathbb{F}}$ (resp. $\mathbf{A}^{\mathbb{F}}_{\leq r} \subset \mathbf{A}^{\mathbb{F}}$) for the moduli space restricted to bundles of rank r (resp. $\leq r$), where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A point of $\mathbf{A}^{\mathbb{F}}$ corresponds to a pair $([E], [\nabla^E])$, where $[E] \to M$ is an equivalence class of vector bundles up to isomorphisms and $[\nabla^E]$ is an equivalence class of metric connections on E up to *gauge-equivalence*; we refer the reader to [20, Chapter 2] for more details on connections and their moduli spaces.

1.2. Isospectral connections

Assume M is endowed with a Riemannian metric g. Given $a = ([E], [\nabla^E]) \in \mathbf{A}^{\mathbb{F}}$, we can form the *Bochner* (or *connection*) *Laplacian* $\Delta_E := (\nabla^E)^* \nabla^E$. While this operator depends on the choice of (E, ∇^E), its unitary conjugacy class depends only on a; note also that Δ_E depends on the metric g since the formal adjoint $(\nabla^E)^*$ of ∇^E does so. It is a formally self-adjoint operator on $L^2(M, E)$ with discrete non-negative spectrum. Let

$$\operatorname{spec}_{L^2}(\Delta_E) := (\lambda_j(\nabla^E))_{j \geq 0}$$

be the sequence of eigenvalues of Δ_E, counted with multiplicities and sorted in increasing order; note that $\lambda_0(\nabla^E) \geq 0$.

Define the *spectrum map* as

$$\mathbf{S} \colon \mathbf{A}^{\mathbb{F}} \ni a \mapsto \operatorname{spec}_{L^2}(\Delta_E) \in \mathbb{R}_{\geq 0}^{\mathbb{Z}_{\geq 0}}. \tag{1.1}$$

The map $\mathbf{S}$ is well defined on the moduli space, that is, it does not depend on a choice of gauge for the connection. Following the celebrated paper of Kac [31], "*Can one hear the shape of a drum?*", one can ask the following question: is the spectrum map (1.1) injective? In other words, does the spectrum of the Bochner Laplacian determine the connection up to gauge-equivalence?

This is the analogous question to the classical inverse spectral problem of recovering a metric g from the knowledge of the spectrum of the usual Hodge Laplacian Δ_g acting on functions. Among hyperbolic surfaces, it is known that the spectrum of the Hodge Laplacian does not determine the metric up to isometries by a result of Vigneras [45]. Nevertheless, Sharafutdinov [42] proved that the spectrum map is locally injective in a neighborhood of a locally symmetric Riemannian space of negative curvature. Apart from negatively curved spaces, other counterexamples were provided by Milnor [38], Sunada [43] using covering spaces, and counterexamples to Kac's isospectral question also exist for piecewise smooth planar domains [25] (for the Dirichlet Laplacian). The problem of characterizing 1-parameter families of isospectral metrics was also studied by various authors and was resolved in negative curvature [29, 19, 41]. It is deeply connected to the *marked length spectrum* conjecture (also known as the Burns-Katok conjecture [11]), see [18, 39, 26]. We refer to [50, 51] for further details about Kac's classical isospectral problem for metrics.

Recall that a metric is said to have *simple length spectrum* if all closed geodesics have different lengths. This is a generic condition with respect to the metric, see [1, 3]. Before stating our result on the injectivity of the spectrum map (1.1) on low-rank vector bundles, we set

$$q_{\mathbb{R}}(n) := q(n), \quad q_{\mathbb{C}}(n) := \tfrac{1}{2}q(n).$$

THEOREM 1.1. – *Let $n \in \mathbb{Z}_{\geq 2}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then for all closed negatively curved Riemannian manifolds (M^{n+1}, g) with simple length spectrum, the spectrum map is injective on $\mathbb{F}$-vector bundles of rank $\leq q_{\mathbb{F}}(n)$:*

$$\mathbf{S}\colon \mathbf{A}^{\mathbb{F}}_{\leq q_{\mathbb{F}}(n)} \to \mathbb{R}_{\geq 0}^{\mathbb{Z}_{\geq 0}}.$$

This seems to be the first result where Kac's inverse spectral problem can be fully solved with a full infinite-dimensional moduli space of geometric objects (here, connections on low-rank vector bundles). In particular, the situation is different from the metric case, where counterexamples are known to exist as discussed above. Earlier results obtained by the authors in [13] could only prove injectivity of the spectrum map (1.1) on small open neighborhoods in the moduli space $\mathbf{A}^{\mathbb{F}}$, but in arbitrary rank: for instance, on a neighborhood of a generic connection, or near flat connections. By means of a trace formula, this is intimately related to the *holonomy inverse problem*, that is, the determination of a connection from the traces of parallel transports along closed geodesics. See Proposition 4.2 below for counterexamples to the injectivity of the holonomy inverse problem in even dimensions.

Also note that, as a (very) particular instance of Theorem 1.1, one can take connections of the form $(M \times \mathbb{C}, d - iA)$ on the trivial complex line bundle, where $A \in C^\infty(M, T^*M)$ is a real-valued 1-form; the Bochner Laplacian $(d - iA)^*(d - iA)$ is also known as the *magnetic Laplacian* in this case. The result then says that one can recover the 1-form A up to gauge (i.e., up to a term of the form $i\varphi^{-1}d\varphi$ for some $\varphi \in C^\infty(M, \mathbb{S}^1)$) from the knowledge of the spectrum of the magnetic Laplacian. In particular, this allows to recover the magnetic field $B := dA$ from the spectrum.