ISSN 0012-9593

quatrième série - tome 58 fascicule 1 janvier-février 2025

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Thomas GAUTHIER & Gabriel VIGNY

The geometric dynamical Northcott and Bogomolov properties

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 494 euros.
Abonnement avec supplément papier :
Europe : 694 €. Hors Europe : 781 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 58, 2025, p. 231 à 273

THE GEOMETRIC DYNAMICAL NORTHCOTT AND BOGOMOLOV PROPERTIES

BY Thomas GAUTHIER AND Gabriel VIGNY

ABSTRACT. – We establish the dynamical Northcott property for polarized endomorphisms of a projective variety over a function field **K** of characteristic zero, and we relate this property to the notion of stability in complex dynamics. This extends previous results of Benedetto, Baker and DeMarco in dimension 1, and of Chatzidakis-Hrushovski in higher dimension. Our proof uses complex dynamics arguments and does not rely on the previous ones.

We first show that, when **K** is the field of rational functions of a normal complex projective variety, the canonical height of a subvariety is the mass of an appropriate bifurcation current and that a marked point is stable if and only if its canonical height is zero. We then establish the geometric dynamical Northcott property characterizing points of height zero in this setting, using a similarity argument. Moving from points to subvarieties, we propose, for polarized endomorphisms, a dynamical version of the geometric Bogomolov conjecture, recently proved by Cantat, Gao, Habegger, and Xie in the original setting of abelian varieties.

RÉSUMÉ. – Nous établissons la propriété dynamique de Northcott pour les endomorphismes polarisés d'une variété projective sur un corps de fonctions **K** de caractéristique nulle, et nous relions cette propriété à la notion de stabilité en dynamique complexe. Cela généralise des résultats de Benedetto, Baker, et DeMarco en dimension 1, et de Chatzidakis-Hrushovski en dimension plus grande. Notre démonstration met en jeu des arguments de dynamique complexe et ne repose pas sur ceux utilisés jusqu'alors.

Dans un premier temps nous prouvons que, lorsque **K** est le corps des fonctions rationnelles d'une variété projective complexe, la hauteur canonique d'une sous-variété coïncide avec la masse d'un courant de bifurcation approprié, et qu'un point marqué est stable si et seulement si sa hauteur canonique est nulle. Nous établissons alors la propriété dynamique géométrique de Northcott caractérisant les points de hauteur nulle dans ce contexte, en utilisant un argument de similarité. En passant de l'étude des points à celle des sous-variétés, nous proposons, pour les endomorphismes polarisés, une version dynamique de la conjecture de Bogomolov géométrique, récemment démontrée par Cantat, Gao, Habegger, et Xie dans le cas des variétés abéliennes.

Both authors are partially supported by the ANR grant Fatou ANR-17-CE40-0002-01.

 doi:10.24033/asens.2605

1. Introduction

1.1. Polarized endomorphisms, functions fields

Fix a field $\mathbf{K}$ of characteristic zero. A *polarized endomorphism* over $\mathbf{K}$ is a triple (X, f, L) where

1. X is a projective variety defined over $\mathbf{K}$, irreducible over $\bar{\mathbf{K}}$;
2. L is an ample line bundle of X which is defined over $\mathbf{K}$, and
3. $f : X \to X$ is an endomorphism defined over $\mathbf{K}$ which is *polarized* by L, i.e., there is an integer $d \geq 2$ such that f^*L is linearly equivalent to $L^{\otimes d}$, denoted $f^*L \simeq L^{\otimes d}$.

The integer d is the *degree* of (X, f, L). A prototypical example is an endomorphism f of a projective space of degree $d \geq 2$, since it satisfies $f^*\mathcal{O}(1) \simeq \mathcal{O}(d) \simeq \mathcal{O}(1)^{\otimes d}$.

In this article, we are interested in the case where $\mathbf{k}$ is an algebraically closed field of characteristic zero and $\mathbf{K} := \mathbf{k}(\mathcal{B})$ is the field of rational functions of an irreducible normal projective $\mathbf{k}$-variety $\mathcal{B}$ of dimension at least one (for our purpose, we can always work with the algebraic closure of $\mathbf{k}$ so we directly assume it is algebraically closed). To such a variety X endowed with the ample line bundle L, we can associate a *model* $(\mathcal{X}, \mathcal{L})$ of (X, L), i.e., a surjective morphism

$$\pi : \mathcal{X} \longrightarrow \mathcal{B}$$

between projective varieties, where $\mathcal{L}$ is a relatively ample line bundle, such that

1. the generic fiber of $\mathcal{X}$ is isomorphic to X,
2. the line bundle L is isomorphic to the restriction of $\mathcal{L}$ to the generic fiber,
3. there exists a dense Zariski open set $\Lambda \subset \mathcal{B}$ over which π is flat.
   We denote $\mathcal{X}_\Lambda := \pi^{-1}(\Lambda)$.

Note that such a model is not unique, there are infinitely many choices (see [11, Chapter 1] for generalities on models of $\mathbf{K}$-schemes). Similarly, Λ is not unique and we will shrink it if necessary. For any $\lambda \in \Lambda(\mathbf{k})$, the fiber $X_\lambda := \pi^{-1}\{\lambda\}$ is a projective $\mathbf{k}$-variety of dimension $\dim X$ and $L_\lambda := \mathcal{L}|_{X_\lambda}$ is an ample line bundle of X_λ. Furthermore, when X is normal, up to replacing $\mathcal{X}$ by its normalization, we can assume $\mathcal{X}$ is normal and X_λ is normal for all $\lambda \in \Lambda$. We call such a normal variety $\mathcal{X}$ a *normal model* of X.

Then, a polarized morphism f defined over $\mathbf{K}$ induces a dominant rational map $f : \mathcal{X} \dashrightarrow \mathcal{X}$ and we can choose a dense Zariski open subset $\Lambda \subset \mathcal{B}$ as above such that in addition

(a) the following diagram commutes

$$\begin{array}{ccc} \mathcal{X} & \overset{f}{\dashrightarrow} & \mathcal{X} \\ & {}_{\pi}\searrow \quad \swarrow_{\pi} & \\ & \mathcal{B}, & \end{array}$$

(b) $f|_{\mathcal{X}_\Lambda} : \mathcal{X}_\Lambda \to \mathcal{X}_\Lambda$ is a morphism,

(c) for any $\lambda \in \Lambda$, if we set $f_\lambda := f|_{X_\lambda}$, then $(X_\lambda, f_\lambda, L_\lambda)$ is a polarized endomorphism over $\mathbf{k}$,

(d) $\mathfrak{f}$ restricted to the generic fiber of $\mathcal{X}$ can be identified with f via the isomorphism between the generic fiber of $\mathcal{X}$ and X given by Property 1 above.

Definition 1.1. – *Let (X, f, L) be a polarized endomorphism over $\mathbf{K} = \mathbf{k}(\mathcal{B})$ where $\mathbf{k}$ is an algebraically closed field of characteristic zero and $\mathbf{K}$ is the field of rational functions of a normal projective $\mathbf{k}$-variety $\mathcal{B}$. A triple $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ satisfying properties* 1.–3. *and* (a)–(d) *above for some dense Zariski open set Λ is called an* algebraic family *of polarized endomorphisms; and Λ is called a regular part. Such a family is a* model *of (X, f, L).*

We will frequently shrink the open set Λ, but we will always assume it is dense and Zariski open. Note that in the article, we will use curly letters $\mathfrak{f}, \mathcal{L}, \mathcal{Z} \dots$ for objects defined on the model $\mathcal{X}$ in order to distinguish them from their counterparts $f, L, Z \dots$ on X. When X is not normal, letting $\mathsf{n} : \hat{X} \to X$ be its normalization, the universal property of normalization implies that f lifts to an endomorphism $\hat{f} : \hat{X} \to \hat{X}$ and, since n is finite, $\mathsf{n}^* L$ is ample, so that $(\hat{X}, \hat{f}, \mathsf{n}^* L)$ still defines a polarized endomorphism.

A *marked point* is a rational section $a : \mathcal{B} \dashrightarrow \mathcal{X}$ whose indeterminacy locus is contained in $\mathcal{B} \setminus \Lambda$, where $\Lambda \subseteq \mathcal{B}$ is a regular part for $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ (when $\mathcal{B}$ is a curve, the indeterminacy locus is empty). In particular, a defines a regular map $a : \Lambda \to \mathcal{X}$ such that $a(\lambda) \in X_\lambda$ for all $\lambda \in \Lambda$. To any subvariety Z of X which is defined over $\mathbf{K}$, we can associate a subvariety $\mathcal{Z}$ such that $\pi|_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{B}$ is flat over a dense Zariski open subset of Λ. A point of $X(\mathbf{K})$ corresponds to a marked point $a : \mathcal{B} \dashrightarrow \mathcal{X}$. We also need the following notion of isotriviality which means, in some sense, that the family does not really depend on the parameter.

Definition 1.2. – *Let (X, f, L) be a polarized endomorphism over $\mathbf{K} = \mathbf{k}(\mathcal{B})$ where $\mathbf{k}$ is an algebraically closed field of characteristic zero and $\mathbf{K}$ is the field of rational functions of a normal projective $\mathbf{k}$-variety $\mathcal{B}$. Let $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ be a model of (X, f, L) and let Λ be a regular part. When X is normal, we say (X, f, L), or equivalently $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$, is* isotrivial *over Λ if, for any $\lambda_0 \in \Lambda(\mathbf{k})$, up to making a base change, there is an isomorphism $\phi : \mathcal{X}_\Lambda \to X_{\lambda_0} \times \Lambda$ such that $\phi \circ \mathfrak{f} = (f_{\lambda_0}, \pi) \circ \phi$ and such that for any $\lambda \in \Lambda(\mathbf{k})$, if $\phi_\lambda := \phi|_{X_\lambda} : X_\lambda \to X_{\lambda_0}$ is the induced isomorphism, then $\phi_\lambda^* L_{\lambda_0} \simeq L_\lambda$.*

In the general case, we say (X, f, L) is isotrivial *over Λ if $(\hat{X}, \hat{f}, \mathsf{n}^* L)$ is, where $\mathsf{n} : \hat{X} \to X$ is its normalization and $\hat{f} : \hat{X} \to \hat{X}$ is the lift of f.*

Remark 1.3. –

1. If (X, f, L) is isotrivial over Λ, so is the polarized variety (X, L) (i.e., in a model, fibers over λ endowed with their polarization are isomorphic) and isotriviality does not depend on the choice of the model.
2. Lemma 3.2 states that the notion of isotriviality is independent of the chosen regular part Λ, and also that a polarized endomorphism (X, f, L) is isotrivial if and only if one of its iterates (X, f^n, L) is. Isotriviality is therefore a dynamically relevant property.
3. When $\mathcal{X} = \mathbb{P}^k \times \mathcal{B}$, the condition $\phi^* L_{\lambda'} \simeq L_\lambda$ is automatically satisfied. Indeed, we have $L = \mathcal{O}_{\mathbb{P}^k}(j)$, for some $j \in \mathbb{N}^*$, so that for any $\lambda \in \Lambda(\mathbf{k})$, we have $L_\lambda \simeq \mathcal{O}_{\mathbb{P}^k}(j)$. In this case, the isotriviality of $(\mathcal{X}, \mathfrak{f}, \mathcal{L})$ implies the existence, for any $\lambda, \lambda' \in \Lambda$, of a linear isomorphism $\phi_\lambda^{\lambda'} : \mathbb{P}^k \to \mathbb{P}^k$ such that $\phi_\lambda^{\lambda'} \circ f_\lambda = f_{\lambda'} \circ \phi_\lambda^{\lambda'}$.