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MEASURE EQUIVALENCE EMBEDDINGS OF FREE GROUPS AND FREE GROUP FACTORS

BY TEY BERENDSCHOT AND STEFAAN VAES

ABSTRACT. – We give a simple and explicit proof that the free group $\mathbb{F}_2$ admits a measure equivalence embedding into any nonamenable locally compact, second countable group G. We use this to prove that every nonamenable locally compact, second countable group admits strongly ergodic actions of any possible Krieger type and admits nonamenable, weakly mixing actions with any prescribed flow of weights.

We also introduce concepts of measure equivalence and measure equivalence embeddings for II_1 factors. We prove that a II_1 factor M is nonamenable if and only if the free group factor $L(\mathbb{F}_2)$ admits a measure equivalence embedding into M. We prove stability of property (T) and the Haagerup property under measure equivalence of II_1 factors.

RÉSUMÉ. – Nous démontrons de façon simple et explicite que le groupe libre $\mathbb{F}_2$ admet un plongement par équivalence mesurée dans tout groupe G localement compact non moyennable et à base dénombrable. Nous déduisons que chacun de ces groupes admet des actions fortement ergodiques avec un type de Krieger arbitraire, et admet des actions non moyennables, faiblement mélangeantes avec un flot des poids prescrit.

Nous introduisons également des concepts d'équivalence mesurée et de plongements par équivalence mesurée pour les facteurs de type II_1. Nous démontrons qu'un facteur de type II_1 M est non moyennable si et seulement si le facteur du groupe libre $L(\mathbb{F}_2)$ admet un plongement par équivalence mesurée dans M. Nous démontrons que la propriété (T) et la propriété de Haagerup sont stables par équivalence mesurée des facteurs de type II_1.

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1. Introduction

The von Neumann-Day problem famously asked whether every nonamenable group contains the free group $\mathbb{F}_2$ as a subgroup. Although this was shown to be false in [16], the problem has a measurable group-theoretic counterpart that does hold. More precisely, it was shown by Gaboriau and Lyons in [8, Theorem 1] that for every countable nonamenable group G, the orbit equivalence relation of the Bernoulli action $G \curvearrowright [0,1]^G$ contains a.e. the orbits of an essentially free action $\mathbb{F}_2 \curvearrowright [0,1]^G$. This implies in particular that $\mathbb{F}_2$ is a measure equivalence (ME) subgroup of G (in the sense of Gromov, see Definition 2.1 for terminology). Gheysens and Monod showed in [9, Theorem 1] that actually every nonamenable locally compact second countable (lcsc) group G contains $\mathbb{F}_2$ as an ME subgroup. Their proof combines the percolation theory methods [8] with the solution of Hilbert's fifth problem on the structure of lcsc groups.

Our first main result is an explicit and simple construction to embed $\mathbb{F}_2$ as an ME subgroup into any nonamenable lcsc group. Because of the explicit nature of our measure equivalence embedding, we can derive several extra properties of weak mixing and strong ergodicity.

As we recall in Section 2.2, a measure equivalence embedding of $\mathbb{F}_2$ into an lcsc group G is defined by a nonsingular action $\mathbb{F}_2 \times G \curvearrowright (\Omega, \gamma)$ whose restrictions to $\mathbb{F}_2$ and to G admit a fundamental domain (in the appropriate sense when G is nondiscrete) and such that $\mathbb{F}_2 \curvearrowright \Omega/G$ admits an equivalent invariant probability measure (the finite covolume condition).

THEOREM A. – *Let G be an lcsc group. Assume that ν is a probability measure on G such that ν is equivalent with the left Haar measure and the convolution operator $\lambda(\nu)$ on $L^2(G)$ has norm less than* $1/3$ *(the existence of which is equivalent to the nonamenability of G). Consider the Bernoulli action $\mathbb{F}_2 \curvearrowright (X, \mu) = (G \times G, \nu \times \nu)^{\mathbb{F}_2}$.*

1. *The action $\mathbb{F}_2 \times G \curvearrowright (\Omega, \gamma) = (X \times G, \mu \times \nu)$ given by*

   $$a \cdot (x, h) = (a \cdot x, (x_e)_1 h), \quad b \cdot (x, h) = (b \cdot x, (x_e)_2 h), \quad g \cdot (x, h) = (x, hg^{-1})$$

   with a, b freely generating $\mathbb{F}_2$ and $g \in G$, is a measure equivalence embedding of $\mathbb{F}_2$ into G.
2. *This ME embedding is weakly mixing: the essentially free action $G \curvearrowright \mathbb{F}_2\backslash\Omega$ is weakly mixing.*
3. *This ME embedding is stably strongly ergodic: $G \curvearrowright \mathbb{F}_2\backslash\Omega$ is strongly ergodic and the diagonal product with any strongly ergodic pmp action $G \curvearrowright (Z, \zeta)$ remains strongly ergodic.*

We prove Theorem A in Section 3.3. Note that by Kesten's theorem (see [1, Théorème 4] in this generality), for any probability measure ν_0 with full support on a nonamenable lcsc group, we have $\|\lambda(\nu_0)\| < 1$, so that a sufficiently high convolution power $\nu = \nu_0^{*k}$ satisfies $\|\lambda(\nu)\| < 1/3$. Therefore, Theorem A proves very explicitly that $\mathbb{F}_2$ admits a measure equivalence embedding into any nonamenable lcsc group G. Note however that for countable groups G, we cannot recover in this way the full strength of the Gaboriau-Lyons theorem [8], which showed that the orbit equivalence relation of $G \curvearrowright [0,1]^G$ contains a.e. the orbits of an essentially free action of $\mathbb{F}_2$.

While the von Neumann-Day problem for groups was solved in the negative in [16], the corresponding problem for II_1 factors remains wide open: does every nonamenable II_1 factor contain a copy of $L(\mathbb{F}_2)$ as a von Neumann subalgebra? While we cannot solve this problem, our simple approach to Theorem A led us to a concept of *measure equivalence (ME) embeddings for II_1 factors* and to proving that $L(\mathbb{F}_2)$ admits a measure equivalence embedding into any nonamenable II_1 factor.

Roughly speaking, an ME embedding of a II_1 factor A into a II_1 factor B is given by a Hilbert A-B-bimodule ${}_A\mathcal{K}_B$ together with the extra data ("the finite covolume condition") of a *finite* von Neumann algebra $Q \subset \mathrm{End}_{A\text{-}B}(\mathcal{K})$ such that the resulting bimodules ${}_A\mathcal{K}_{Q^{\mathrm{op}}}$ and ${}_Q\mathcal{K}_B$ are both coarse and the latter is also finitely generated. If both are finitely generated, we get the concept of a *measure equivalence* of A and B. The precise definition is given in 5.1.

Our concept of ME embeddings for finite von Neumann algebras is compatible with the concept of ME embeddings for groups: we prove in Proposition 5.5 that an ME embedding of a countable group Γ into a countable group Λ gives rise, in a canonical way, to an ME embedding of $L(\Gamma)$ into $L(\Lambda)$.

On the one hand, our notion of measure equivalence (embedding) for II_1 factors is strong enough to capture qualitative properties of II_1 factors: if A admits an ME embedding into B, then amenability or the Haagerup approximation property are inherited (see Proposition 5.7) from B to A, and Kazhdan's property (T) is preserved under measure equivalence (Proposition 5.9). On the other hand, our notion of measure equivalence embedding for II_1 factors is sufficiently flexible to obtain the following positive variant of the von Neumann-Day problem for II_1 factors.

THEOREM B. – *A II_1 factor M is nonamenable if and only if $L(\mathbb{F}_2)$ admits a measure equivalence embedding into M.*

Moreover, we prove in Proposition 5.10 that in general, measure equivalence of II_1 factors is strictly weaker than commensurability (i.e., the existence of a finite index bimodule). Altogether these results show that our new concept of measure equivalence (embeddings) is quite natural.

Going back to groups, every ME embedding of an lcsc group H into an lcsc group G allows to induce nonsingular H-actions to nonsingular G-actions. In particular, the concrete ME embeddings of Theorem A allow to induce nonsingular actions of the free group to nonsingular actions of an arbitrary nonamenable lcsc group G. Since the ME embeddings in Theorem A have several extra properties, we can then prove at the end of Section 4 the following result, whose third point answers a question posed by Danilenko in [5] because strongly ergodic actions on nonatomic spaces are in particular nonamenable.

THEOREM C. – *Let G be a nonamenable lcsc group. Let $\mathbb{R} \curvearrowright (Z,\zeta)$ be an ergodic nonsingular flow and $\lambda \in (0,1]$.*

1. *There exists an essentially free, weakly mixing, nonsingular action $G \curvearrowright (X,\mu)$ that is nonamenable in the sense of Zimmer and such that the crossed product $L^\infty(X) \rtimes G$ has flow of weights $\mathbb{R} \curvearrowright Z$.*