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FINITELY PRESENTED SIMPLE LEFT-ORDERABLE GROUPS IN THE LANDSCAPE OF RICHARD THOMPSON'S GROUPS

BY JAMES HYDE AND YASH LODHA

ABSTRACT. – We construct the first examples of finitely presented simple groups of orientation-preserving homeomorphisms of the real line. Our examples are also of type F_∞, have infinite geometric dimension, and admit a nontrivial homogeneous quasimorphism (and hence have infinite commutator width).

RÉSUMÉ. – Nous construisons les premiers exemples de groupes simples de présentation finie et ordonnables à gauche. Nos exemples satisfont également la propriété de finitude plus forte F_∞, ont une dimension géométrique infinie, et admettent un quasimorphisme homogène non trivial.

1. Introduction

A natural line of investigation in modern group theory seeks to understand the landscape of *finitely presented, infinite, simple groups*. The first such examples were discovered by Richard Thompson in 1965, and are referred to in the literature as Thompson's groups T and V [10]. In the subsequent decades various "Thompson-like" examples of finitely presented infinite simple groups emerged in the work of Brown [4], Higman [17], Scott [26], Stein [28], Brin [3], Rover-Nekrashevich [27] and more recently, among others, in [21, 8]. A feature of all these examples, including T and V, is that they are not torsion-free. In particular, they cannot act faithfully on the line by orientation-preserving homeomorphisms. In their seminal article [7], Burger and Mozes constructed the first finitely presented simple torsion-free groups. They obtained a family of such groups, emerging as lattices in products of automorphism groups of regular trees. For each of them it remains unknown whether it admits a nontrivial action by homeomorphisms on the real line. Finitely presented infinite simple groups were also constructed within the realm of non-affine Kac-Moody groups [12]. However, the groups in [12] are not torsion-free. In this article we construct the first family of finitely presented left-orderable simple groups, proving:

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THEOREM 1.1. – *There exist finitely presented (and type $\mathbf{F}_\infty$) simple groups of orientation-preserving homeomorphisms of* $\mathbf{R}$*. Equivalently, there exist finitely presented (and type $\mathbf{F}_\infty$) simple left-orderable groups.*

Whether finitely generated infinite simple groups of homeomorphisms of $\mathbf{R}$ exist was a longstanding open problem [20, Problem 16.50]. This was solved by the authors in [18], where we exhibited continuum many (up to isomorphism) examples. Subsequently, Matte Bon and Triestino in [24] provided a more conceptual generalization of our construction and new classes of examples emerged in the work of the authors with Rivas in [19]. However, none of these examples are finitely presented since they emerge naturally as nontrivial limits in the Grigorchuk space of marked groups. Theorem 1.1 is proved by means of the following new construction.

DEFINITION 1.2. – For $n \geq 2$, we define $\Gamma_n \leq \mathrm{Homeo}^+(\mathbf{R})$ as the group of homeomorphisms $f \in \mathrm{Homeo}^+(\mathbf{R})$ satisfying the following:

1. f is piecewise linear with breakpoints in $\mathbf{Z}[\frac{1}{n(n+1)}]$, and $\mathbf{Z}[\frac{1}{n(n+1)}] \cdot f = \mathbf{Z}[\frac{1}{n(n+1)}]$. (A *breakpoint* is a point where the left and right derivatives do not coincide.)
2. f commutes with the translation $t \mapsto t+1$.
3. For each $x \in \mathbf{R} \setminus \mathbf{Z}[\frac{1}{n(n+1)}]$, there exist (unique) $i, j \in \mathbf{Z}$ such that $x \cdot f' = n^i (n+1)^j$ and one has:

$$i - j = |\mathbf{Z} \cap (x, x \cdot f)| \text{ if } x \leq x \cdot f \qquad i - j = -|\mathbf{Z} \cap (x \cdot f, x)| \text{ if } x > x \cdot f.$$

THEOREM 1.3. – *For each $n \geq 2$, the group $Q_n = [\Gamma_n, \Gamma_n]$ is a finitely presented (and type $\mathbf{F}_\infty$) simple group of orientation-preserving homeomorphisms of* $\mathbf{R}$.

First observations. – We supply some basic facts about our groups Γ_n, Q_n.

PROPOSITION 1.4. – *For each $n \in \mathbf{N}, n \geq 2$, let Γ_n denote the set of homeomorphisms from Definition 1.2 and fix $\eta_n = n(n+1)$. Then the following holds:*

1. *Γ_n is a subgroup of* $\mathrm{Homeo}^+(\mathbf{R})$.
2. *The stabilizer of* 0 *in Γ_n is isomorphic to the Higman-Thompson group F_{η_n}. In particular, Γ_n, Q_n have infinite geometric dimension.*
3. *Γ_n, Q_n embed in the group of piecewise linear orientation-preserving homeomorphisms of* $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$.
4. *Γ_n, Q_n do not have Kazhdan's property (T), and they contain nonabelian free subgroups.*

We recall the definition of the Higman-Thompson groups F_n, for each $n \in \mathbf{N}, n \geq 2$. F_n is the group of orientation-preserving piecewise linear homeomorphisms $f : [0,1] \to [0,1]$ whose slopes lie in $\{n^m \mid m \in \mathbf{Z}\}$ and breakpoints lie in $\mathbf{Z}[\frac{1}{n}]$. Denote this as the *standard action* of F_n on $[0,1]$.

We define the 1-*periodic action* of F_n as the unique embedding $F_n \leq \mathrm{Homeo}^+(\mathbf{R})$ which commutes with integer translations, fixes $\mathbf{Z}$ pointwise, and whose restriction to $[0,1]$ is the standard action of F_n.

Proof of Proposition 1.4. – We will need the following definitions. For $x, y \in \mathbf{R}$, we define $(x, y)_{\mathbf{Z}}$ as:

$$(x, y)_{\mathbf{Z}} := |(x, y) \cap \mathbf{Z}| \text{ if } x \leq y \qquad (x, y)_{\mathbf{Z}} := -|(y, x) \cap \mathbf{Z}| \text{ if } y < x$$

This satisfies the relation $(x, y)_{\mathbf{Z}} + (y, z)_{\mathbf{Z}} = (x, z)_{\mathbf{Z}}$ for all $x, y, z \in \mathbf{R} \setminus \mathbf{Z}$. For each $n \in \mathbf{N}, n \geq 2$, fix $\Omega_n := \{n^{k_1}(n+1)^{k_2} \mid k_1, k_2 \in \mathbf{Z}\}$ as the multiplicative subgroup of $\mathbf{R}_{>0}$.

We define the homomorphism $\alpha : \Omega_n \to \mathbf{Z}$ as $\alpha(\gamma) = k_1 - k_2$ for $\gamma = n^{k_1}(n+1)^{k_2} \in \Omega_n$.

Part (1). – Γ_n clearly contains the identity. We show that for $f, g \in \Gamma_n$, $fg \in \Gamma_n$, and leave the proof that $f^{-1} \in \Gamma_n$ as an exercise. The element fg clearly satisfies conditions (1), (2) of Definition 1.2. Condition (3) for fg is equivalent to the assertion that for all $x \in \mathbf{R} \setminus \mathbf{Z}[\frac{1}{\eta_n}]$, $(x, x \cdot fg)_{\mathbf{Z}} = \alpha(x \cdot (fg)')$. Since $f, g \in \Gamma_n$, we know that $(x, x \cdot f)_{\mathbf{Z}} = \alpha(x \cdot f')$ and $(x, x \cdot g)_{\mathbf{Z}} = \alpha(x \cdot g')$. The chain rule implies that $\alpha(x \cdot (fg)') = \alpha(x \cdot f') + \alpha((x \cdot f) \cdot g') = (x, x \cdot f)_{\mathbf{Z}} + (x \cdot f, x \cdot fg)_{\mathbf{Z}} = (x, x \cdot fg)_{\mathbf{Z}}$.

Part (2). – Each element of the 1-periodic action of F_{η_n} satisfies Definition 1.2, hence lies in Γ_n. Any element $f \in \Gamma_n$ that fixes 0 must pointwise fix $\mathbf{Z}$, due to condition (2) of Definition 1.2. Indeed, it follows from Definition 1.2 that f is an element in the 1-periodic action of F_{η_n}. Finally, it is a standard fact that F'_{η_n} contains copies of $\bigoplus_{\mathbf{Z}} \mathbf{Z}$, which has infinite geometric dimension (a feature inherited by overgroups).

Parts (3), (4). – By Definition 1.2, the groups Γ_n, Q_n have the property that each element commutes with all integer translations, yet no integer translation is contained in Γ_n. It follows that both actions descend to faithful actions by orientation-preserving piecewise linear homeomorphisms of $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$. It follows from the main theorems in [13] and [22] that Γ_n does not have Kazhdan's property (T). It follows from a standard "ping-pong argument" for dense subgroups of $\mathrm{Homeo}^+(\mathbf{S}^1)$ [23] (see also Theorem 2.3.2 in [25]) that they contain nonabelian free subgroups. □

An alternative action. – A remark of James Belk after the first version of the article appeared provides an elegant alternative action of Γ_2 on $\mathbf{R}_{>0}$.

REMARK 1.5. – Consider the piecewise linear, orientation reversing homeomorphism $\kappa : \mathbf{R}_{>0} \to \mathbf{R}$ that maps $[2^k, 2^{k+1}] \mapsto [-(k+1), -k]$ linearly for each $k \in \mathbf{Z}$. Note that κ conjugates $t \mapsto 2t$ to $t \mapsto t - 1$. In fact, κ^{-1} conjugates the given action of Γ_2 on $\mathbf{R}$ to the group consisting of all piecewise linear maps $f \in \mathrm{Homeo}^+(\mathbf{R}_{>0})$ satisfying: the slope $x \cdot f'$, whenever defined, lies in $\{6^n \mid n \in \mathbf{Z}\}$, the breakpoints of f lie in $\mathbf{Z}[\frac{1}{6}] \cap \mathbf{R}_{\geq 0}$, and f commutes with $t \mapsto 2t$.

An anonymous referee pointed out that we can also define $\Gamma_n, n \geq 2$ in this fashion, as follows.

REMARK 1.6. – Fix $n \in \mathbf{N}, n \geq 2$, and for each $k \in \mathbf{Z}$, let $u_k = \frac{n^{-k}}{n-1}$. Note that $u_k \in \mathbf{Z}[\frac{1}{n}] + \frac{1}{(n-1)}$ and $u_{k-1} - u_k = n^{-k}$ for all $k \in \mathbf{Z}$. We define a piecewise linear decreasing map $\kappa : \mathbf{R} \to \mathbf{R}_{>0}$ as follows. For each $k \in \mathbf{Z}$, $k \cdot \kappa = u_k$ and κ is affine on each $[k-1, k]$ with slope $-n^{-k}$. Then κ conjugates Γ_n to the group of piecewise affine homeomorphisms of the interval $\mathbf{R}_{>0}$ with breakpoints in the additive coset $\mathbf{Z}[\frac{1}{n(n+1)}] + \frac{1}{(n-1)}$, whose slopes are