ISSN 0012-9593

*quatrième série - tome 58 fascicule 2 mars-avril 2025*

# *ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE*

Gavril FARKAS & Eric LARSON

*The minimal resolution property for points on general curves*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique





**Rédaction / *Editor***

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88.
Email : `annales@ens.fr`



**Édition et abonnements / *Publication and subscriptions***

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : `abonnements@smf.emath.fr`

**Tarifs**

Abonnement électronique : 494 euros.
Abonnement avec supplément papier :
Europe : 694 €. Hors Europe : 781 € ($ 985). Vente au numéro : 77 €.





ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

*Ann. Scient. Éc. Norm. Sup.*
4$^{e}$ série, t. 58, 2025, p. 433 à 462

# THE MINIMAL RESOLUTION PROPERTY FOR POINTS ON GENERAL CURVES

BY Gavril FARKAS AND Eric LARSON

ABSTRACT. – We determine when the resolution of a general set of points on a general curve satisfies the Minimal Resolution property. In particular, we completely determine the shape of the minimal resolution of general sets of points on a general curve $C \subseteq \mathbb{P}^r$ of degree $d \geq 2r$. Our methods also provide a proof (valid in arbitrary characteristic) of the strong version of Butler's conjecture on the stability of syzygy bundles on a general curve of every genus $g > 2$ in projective space, as well as of the strong (Frobenius) semistability in positive characteristic of the syzygy bundle of a general curve $C \subseteq \mathbb{P}^r$ in the range $d \geq 2r$.

RÉSUMÉ. – Nous déterminons quand la résolution d'un ensemble général de points sur une courbe générale satisfait la propriété de résolution minimale. En particulier, nous déterminons complètement la forme de la résolution minimale d'ensembles généraux de points sur une courbe générale $C$ dans $\mathbb{P}^r$ de degré $d \geq 2r$. Nos méthodes fournissent également une preuve (valable en caractéristique arbitraire) de la version forte de la conjecture de Butler sur la stabilité des fibrés de syzygies sur une courbe générale de genre $g > 2$ quelconque dans l'espace projectif, ainsi que de la semistabilité forte en caractéristique positive du fibré de syzygies d'une courbe générale $C$ dans $\mathbb{P}^r$ dans l'intervalle $d \geq 2r$.

## 1. Introduction

For an embedded projective variety $X \subseteq \mathbb{P}^r$ one can ask whether the minimal free resolution of a general set of (sufficiently many) points of $X$ is determined by the geometry of $X$. We shall provide an essentially complete solution to this question for general curves in projective space.

Setting $S := k[x_0, \ldots, x_r]$, where $k$ is an algebraically closed field of arbitrary characteristic, we recall that a finitely generated graded $S$-module $M$ has a minimal free resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow \cdots \leftarrow F_i \leftarrow \cdots,$$

where $F_i = \bigoplus_{j>0} S(-i-j)^{b_{i,j}(M)}$. The graded Betti numbers $b_{i,j}(M) = \dim_k \mathrm{Tor}_i^S(M,k)_{i+j}$ are uniquely determined and can be computed via Koszul cohomology. The Betti diagram of $M$ is obtained by placing the entry $b_{i,j}(M)$ in the $i$-th column and $j$-th row.


doi:10.24033/asens.2609
ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE

Let $X \subseteq \mathbb{P}^r$ be an embedded projective variety and denote by $P_X(t)$ its Hilbert polynomial. We fix a general subset $\Gamma \subseteq X$ of $\gamma$ points and require that $\gamma \geq P_X(m)$, where $m = \mathrm{reg}(X)$ is the Castelnuovo-Mumford regularity of $X$. If $u \geq \mathrm{reg}(X) + 1$ is the integer determined by the condition $P_X(u-1) \leq \gamma < P_X(u)$, it has been shown in [13] that the Betti diagram of $\Gamma$ is obtained from the Betti diagram of $X$ by adding two rows indexed by $u-1$ and $u$, that is, $b_{i,j}(\Gamma) = b_{i,j}(X)$ for $j \leq u-2$, whereas $b_{i,j}(\Gamma) = 0$ for $j \geq u+1$. The Minimal Resolution property (MRP) for $X$ is the statement

$$b_{i,u}(\Gamma) \cdot b_{i+1,u-1}(\Gamma) = 0, \tag{1}$$

for all $\gamma \geq \mathcal{P}_X\big(\mathrm{reg}(X)\big)$ as described above and for all $i \geq 0$, see [21, 13]. Since the differences $b_{i,u}(\Gamma) - b_{i+1,u}(\Gamma)$ are explicitly determined by the Hilbert polynomial of $X$, the Minimal Resolution property for $X$ determines entirely the Betti diagram of $\Gamma$ and it implies that the Betti numbers of $\Gamma$ are as small as the geometry (that is, the Hilbert polynomial) of $X$ allows.

The Minimal Resolution property (under the name of Minimal Resolution conjecture) has been intensely studied when the variety in question is the projective space. In that case, the resolution of a general set $\Gamma \subseteq \mathbb{P}^r$ of sufficiently many $\gamma$ points has only two non-trivial rows, indexed $u-1$ and $u$ respectively, and MRP implies that the resolution is *natural*, that is, at each step only one non-trivial Betti number appears. MRP is known to hold for $r \leq 4$, as well as for a very large number of points in any projective space, due to work of Hirschowitz and Simpson [15]. However, counterexamples to MRP in any projective space $\mathbb{P}^r$, where $r \geq 6$ and $r \neq 9$, have been found by Eisenbud, Schreyer, Popescu and Walter, see [11, 12]. The question has also been studied when $X \subseteq \mathbb{P}^3$ is a smooth surface of small degree, see [5], or for a $K3$ surface in [1]. MRP has been proved to hold for all canonical curves, see [13], and linked to important questions on the moduli space of vector bundles on curves.

We now focus on the case when $X = C$ is a smooth curve embedded by a (not necessarily complete) linear series $\ell = (L, V) \in G^r_d(C)$. Basic Brill-Noether theory ensures that when $\rho(g,r,d) = g - (r+1)(g-d+r) \geq 0$ the stack $\mathcal{G}^r_d$ parametrizing such pairs $(C, \ell)$ has a unique component dominating the moduli space $\mathcal{M}_g$. A pair $[C, \ell]$ corresponding to a (general) point of this component is referred to as a *(general) Brill-Noether (BN)* curve. It was pointed out in [13] via vector bundle techniques that property (1) fails for every curve $C \subseteq \mathbb{P}^r$ for certain values of $i$ when $d$ is large with respect to $g$. Common to these counterexamples is that they occur in the range $d < 2r$ (see also (6) for further explanations). Confirming the expectation, already formulated in [1], that MRP holds outside this range is the main result of this paper.

THEOREM 1.1. – *Let $C \subseteq \mathbb{P}^r$ be a general Brill-Noether curve of genus $g \geq 1$ and degree $d \geq 2r$. Then the Minimal Resolution property holds for $C$.*

To spell out the statement of Theorem 1.1, if $C \subseteq \mathbb{P}^r$ is a general Brill-Noether curve of degree $d \geq 2r$ and $\Gamma \subseteq C$ is a general set of $\gamma \geq d \cdot \mathrm{reg}(C) + 1 - g$ points, setting

$$u := 1 + \Big\lfloor \frac{\gamma + g - 1}{d} \Big\rfloor,$$

the Betti diagram of $\Gamma$ is obtained by adding to the Betti diagram of $C$ precisely the rows indexed by $u-1$ and $u$ respectively. The entries in these rows are explicitly given as follows:

$$b_{i,u}(\Gamma) = 0 \text{ for } i \le r\left(1 - \left\{\frac{\gamma+g-1}{d}\right\}\right) \quad \text{and}$$

$$b_{i,u}(\Gamma) = d\binom{r}{i}\left(\frac{i}{r} + \left\{\frac{\gamma+g-1}{d}\right\} - 1\right) \quad \text{for } i > r\left(1 - \left\{\frac{\gamma+g-1}{d}\right\}\right).$$

Here $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of a number $x$.

| 1 | ... | $i$ | $i+1$ | ... |
|---|---|---|---|---|
| $b_{1,1}(C)$ | ... | $b_{i,1}(C)$ | $b_{i+1,1}(C)$ | ... |
| ... | ... | ... | ... | ... |
| $b_{1,u-2}(C)$ | ... | $b_{i,u-2}(C)$ | $b_{i+1,u-2}(C)$ | ... |
| $b_{1,u-1}(\Gamma)$ | ... | $b_{i,u-1}(\Gamma)$ | $b_{i+1,u-1}(\Gamma)$ | ... |
| $b_{1,u}(\Gamma)$ | ... | $b_{i,u}(\Gamma)$ | $b_{i+1,u}(\Gamma)$ | ... |
| 0 | ... | 0 | 0 | ... |

TABLE 1. The Betti table of a general set $\Gamma \subseteq C$ of $\gamma \gg 0$ points

A version of Theorem 1.1 with a much more restrictive bound for $d$ has been established in [1]. In order to clarify the relevance of the condition $d \ge 2r$ to MRP, we recall the Koszul-theoretic interpretation of the Betti numbers of $\Gamma$. If $\ell = (L, V) \in G^r_d(C)$ is the linear system inducing the embedding $C \subseteq \mathbb{P}^r$, the kernel vector bundle $M_V$ is constructed via the exact sequence

$$0 \longrightarrow M_V \longrightarrow V \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0.$$

Using standard Koszul cohomology arguments [13, Proposition 1.6], one finds

$$b_{i+1,u-1}(\Gamma) = h^0\Big(C, \bigwedge^i M_V \otimes \mathcal{I}_{\Gamma/C}(u)\Big) \text{ and } b_{i,u}(\Gamma) = h^1\Big(C, \bigwedge^i M_V \otimes \mathcal{I}_{\Gamma/C}(u)\Big). \tag{2}$$

Since $\mathrm{rk}(M_V) = r$ and $\deg(M_V) = -d$, by Riemann-Roch one computes

$$b_{i+1,u-1}(\Gamma) - b_{i,u}(\Gamma) = \chi\Big(C, \bigwedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(u)\Big) = \binom{r}{i}\Big(-\frac{id}{r} + du - \gamma + 1 - g\Big),$$

which explains how the $u$-th row of the Betti diagram of $\Gamma$ determines its $(u-1)$-st row. Using (2) it is easy to show that $C \subseteq \mathbb{P}^r$ satisfies the Minimal Resolution property if and only if the kernel bundle $M_V$ verifies the following generic vanishing conditions

$$H^0\Big(C, \bigwedge^i M_V \otimes \xi\Big) = 0, \tag{3}$$

for each $i = 0, \dots, r$ and for a general line bundle $\xi \in \mathrm{Pic}^{g-1+\lfloor \frac{id}{r} \rfloor}(C)$, see also [13, Corollary 1.8]. Note that the degree of $\xi$ is chosen maximally in such a way that the vanishing (3) could possibly hold, thus the statement (3), if true, is sharp. It turns out that (3) is related