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MODULI OF HYBRID CURVES I: VARIATIONS OF CANONICAL MEASURES

BY OMID AMINI AND NOEMA NICOLUSSI

To Maryam Mirzakhani, with admiration for her sense of beauty in mathematics.

ABSTRACT. – The present paper is the first in a series devoted to the study of asymptotic geometry of Riemann surfaces and their moduli spaces.

We introduce the moduli space of hybrid curves as a new compactification of the moduli space of curves, refining the one obtained by Deligne and Mumford. This is the moduli space for multiscale geometric objects which mix complex and higher rank tropical and non-Archimedean geometries, reflecting both discrete and continuous features.

We define canonical measures on hybrid curves which combine and generalize Arakelov-Bergman measures on Riemann surfaces and Zhang measures on metric graphs.

We then show that the universal family of canonically measured hybrid curves over this moduli space varies continuously. This provides a precise link between the non-Archimedean Zhang measure and variations of Arakelov-Bergman measures in families of Riemann surfaces, answering a question which has been open since the pioneering work of Zhang on admissible pairing in the nineties.

RÉSUMÉ. – Le présent article est le premier d'une série de travaux consacrés à l'étude de la géométrie asymptotique des surfaces de Riemann et de leurs espaces de modules.

Nous introduisons l'espace de modules des courbes hybrides comme une nouvelle compactification de l'espace de modules des courbes, raffinant celle construite par Deligne et Mumford. Il s'agit de l'espace de modules pour les objets géométriques multi-échelles qui mélangent la géométrie complexe et la géométrie tropicale et non archimédienne de rang supérieur, reflétant à la fois des caractéristiques discrètes et continues.

Nous définissons des mesures canoniques sur les courbes hybrides qui généralisent les mesures d'Arakelov-Bergman sur les surfaces de Riemann et les mesures de Zhang sur les graphes métriques.

Nous montrons ensuite que la famille universelle de courbes hybrides munies de leurs mesures canoniques est une famille continue d'espaces mesurables au-dessus de cet espace de modules hybride. Ce résultat fournit un lien précis entre la mesure de Zhang non archimédienne et les variations des mesures d'Arakelov-Bergman dans les familles de surfaces de Riemann, répondant ainsi à une question ouverte depuis les travaux pionniers de Zhang sur l'accouplement admissible dans les années quatre-vingt-dix.

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1. Introduction

In this paper we study *canonical measures* on Riemann surfaces and their *tropical and hybrid limits*.

By canonical measure on a Riemann surface we mean the Arakelov-Bergman measure μ_{Ar} defined in terms of holomorphic one-forms. For a compact Riemann surface S of positive genus g, this is the positive density measure of total mass g on S given by

$$\mu_{\mathrm{Ar}} := \frac{\mathrm{i}}{2}\sum_{j=1}^{g} \omega_j \wedge \bar{\omega}_j,$$

where $\mathrm{i} = \sqrt{-1}$ and $\omega_1, \dots, \omega_g$ form an orthonormal basis for the space of holomorphic one-forms on S, with respect to the Hermitian inner product

$$\langle \eta_1, \eta_2 \rangle := \frac{\mathrm{i}}{2}\int_S \eta_1 \wedge \bar{\eta}_2$$

for pairs of holomorphic one-forms η_1, η_2 on S.

Let $\mathscr{M}_g$ be the moduli space of curves of genus g and let $\overline{\mathscr{M}}_g$ be its Deligne-Mumford compactification, consisting of stable curves of genus g. The question we address in this paper is the following:

QUESTION 1.1. – *Consider a sequence of smooth compact Riemann surfaces S_j of genus g such that the corresponding points s_j of $\mathscr{M}_g$ converge to a point in $\overline{\mathscr{M}}_g$. Let μ_j^{can} be the canonical measure of S_j. What is the asymptotic behavior of the measures μ_j^{can}?*

Since the measures μ_j^{can} live on distinct surfaces, the first problem to handle consists in giving a precise mathematical meaning to the question. Moreover, once this has been taken care of, the answer to the question appears to be sensitive to the *speed* and *direction* of the convergence of the sequence of points (s_j). Formalizing these points leads to the definition of new geometric objects called *hybrid curves* and their *moduli spaces*.

Although discovered in context with canonical measures, hybrid curves and their moduli spaces provide a framework in which several other questions can be answered as well, see Section 1.5 for a brief outlook on further developments.

1.1. Hybrid curves

Let s_∞ be the limit of (s_j) and let S_∞ be the stable Riemann surface corresponding to s_∞. Denote by $G = (V, E)$ the dual graph of S_∞: its vertices are in bijection with the irreducible components of S_∞ and its edges are in bijection with the nodes (singular points) of S_∞. If an edge e corresponds to a node p, then its endpoints are given by the two components of S_∞ (possibly identical, if e is a loop) containing p. The genera of components of S_∞ define a *genus function* $\mathfrak{g}\colon V \to \mathbb{N}$. The pair $(G, \mathfrak{g})$ is called the *stable dual graph* of S_∞.

With the help of the stable dual graph, it will be possible to capture additional information on the speed and direction of convergence of the sequence.

We can view the speed of convergence as a way to distinguish the *relative order of appearance of the singular points* in the limit, when the sequence $S_1, S_2, \dots$ approaches the stable Riemann surface S_∞. This leads to an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of the edge set E, where π_1 are the edges corresponding to the *fastest appearing nodes*, π_2 are the ones which are

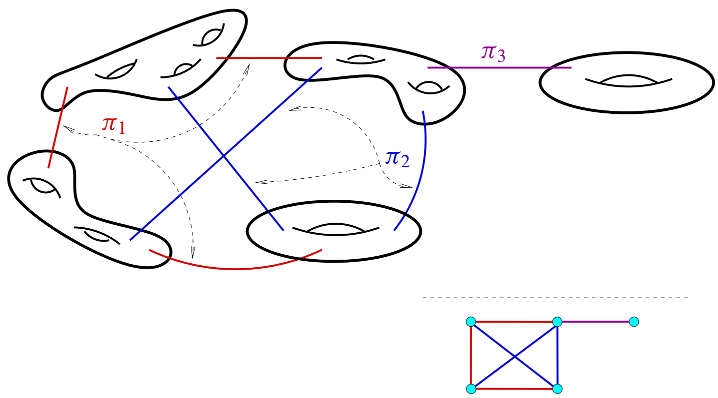

FIGURE 1. An example of a hybrid curve of rank three. The graph of the underlying stable Riemann surface has five vertices and seven edges. Its edges are partitioned into three sets π_1, π_2, and π_3.

fastest among the remaining nodes, and so on. We will call the ordered partition π a *layering* of the graph, the elements of π the *layers*, and the graph G endowed with the layering a *layered graph*.

Capturing the direction corresponds to choosing edge length functions

$$\ell_1 \colon \pi_1 \to \mathbb{R}_+, \dots, \ell_r \colon \pi_r \to \mathbb{R}_+$$

on the layers, each of them well-defined up to multiplication by a positive scalar. Here and everywhere else, $\mathbb{R}_+$ denotes the set of strictly positive real numbers.

We have now arrived at the data underlying the definition of a hybrid curve:

— a stable curve S with stable dual graph $G = (V, E, \mathfrak{g})$ and

— an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of E, for some $r \in \mathbb{N}$, and

— an edge length function $\ell \colon E \to \mathbb{R}_+$.

Given the triple (S, ℓ, π), we define the associated *layered metrized complex* as the metrized complex $\mathcal{MC}$ from [5], obtained as the *metric realization* of (S, ℓ), and enriched with the data of the layering π.

We recall that $\mathcal{MC}$ is obtained by taking first the normalization $\tilde{S}$ of S, which is by definition the disjoint union of the (normalization of the) irreducible components of S, then taking an interval $\mathcal{I}_e$ of length ℓ_e for each edge $e \in E$ associated to a node p_e in S, and finally gluing the two extremities of $\mathcal{I}_e$ to the two points in the normalization $\tilde{S}$ corresponding to the node p_e in S (see Figure 1).

We then define the following *conformal equivalence relation* (*at infinity*) on layered metrized complexes. Two layered metrized complexes $\mathcal{MC}$ and $\mathcal{MC}'$ are conformally equivalent if there exists an isomorphism of the semistable curves underlying $\mathcal{MC}$ and $\mathcal{MC}'$ so that under this isomorphism, they have the same underlying graph $G = (V, E)$ (this will be automatic), the same ordered partition $\pi = (\pi_1, \dots, \pi_r)$ on E, and for each layer π_j, $j = 1, \dots, r$ of π, there is a positive number $\lambda_j > 0$ such that

$$\ell|_{\pi_j} = \lambda_j \, \ell'|_{\pi_j}$$