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NEW CASES OF p-ADIC UNIFORMIZATION

by

Stephen Kudla & Michael Rapoport

To G. Laumon on his 60th birthday

Abstract. — We prove a Cherednik style p-adic uniformization theorem for Shimura varieties associated to certain groups of unitary similitudes of size two over totally real fields. Our basic tool is the alternative modular interpretation of the Drinfeld p-adic halfplane of our earlier paper [**16**].

***Résumé* (Nouveaux cas d'uniformisation p-adique).** — On démontre un théorème d'uniformisation p-adique à la Cherednik pour les variétés de Shimura associées à certains groupes de similitudes unitaires de rang deux sur des corps totalement réels. L'outil principal est notre interprétation modulaire alternative du demi-plan de Drinfeld p-adique dans [**16**].

1. Introduction

The subject matter of p-adic uniformization of Shimura varieties starts with Cherednik's paper [**5**] in 1976, although a more thorough historical account would certainly involve at least the names of Mumford and Tate. Cherednik's theorem states that the Shimura curve associated to a quaternion algebra B over a totally real field F which is split at precisely one archimedean place v of F (and ramified at all other archimedean places), and is ramified at a non-archimedean place w of residue characteristic p admits p-adic uniformization by the Drinfeld halfplane associated to F_w, provided that the level structure is prime to p. In adelic terms, this theorem may be formulated more precisely as follows.

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Let C be an open compact subgroup of $(B \otimes_F \mathbb{A}_F^\infty)^\times$ of the form

$$C = C^w \cdot C_w,$$

where $C_w \subset (B \otimes_F F_w)^\times$ is maximal compact and $C^w \subset (B \otimes_F \mathbb{A}_F^{\infty,w})^\times$. Let $\mathcal{S}_C$ be the associated Shimura curve. It has a canonical model over F and its set of complex points, for the F-algebra structure on $\mathbb{C}$ given by v, has a complex uniformization

$$\mathcal{S}_C(\mathbb{C}) = B^\times \backslash \big[\mathbf{X} \times (B \otimes_F \mathbb{A}_F^\infty)^\times / C\big],$$

where $\mathbf{X} = \mathbb{C} \smallsetminus \mathbb{R}$, which is acted on by $(B \otimes_F F_\mathbb{R})^\times$ *via* a fixed isomorphism $B_v^\times \simeq \mathrm{GL}_2(\mathbb{R})$.

Cherednik's theorem states that, after extending scalars from F to $\bar{F}_w$, there is an isomorphism of algebraic curves over $\bar{F}_w$,

$$\mathcal{S}_C \otimes_F \bar{F}_w \simeq \big(\bar{B}^\times \backslash \big[\Omega^2_{F_w} \times (B \otimes_F \mathbb{A}_F^\infty)^\times / C\big]\big) \otimes_{F_w} \bar{F}_w, \tag{1.1}$$

where $\bar{B}$ is the quaternion algebra over F, with the same invariants as B, except at v and w, where they are interchanged. Here $\Omega^2_{F_w}$ is the rigid-analytic space $\mathbb{P}^1_{F_w} \smallsetminus \mathbb{P}^1(F_w)$ over F_w (*Drinfeld's halfspace*). This isomorphism is to be interpreted as follows.

The rigid-analytic space $\bar{B}^\times \backslash \big[\Omega^2_{F_w} \times (B \otimes_F \mathbb{A}_F^\infty)^\times / C\big]$ corresponds to a unique projective algebraic curve over Spec F_w under the GAGA functor. In the right hand side of (1.1), we implicitly replace the rigid-analytic space by this projective scheme; extending scalars, we obtain a projective algebraic curve over $\bar{F}_w$. The statement of Cherednik's theorem is that there exists an isomorphism between these two algebraic curves over $\bar{F}_w$.

Drinfeld [**8**] gave a moduli-theoretic proof of Cherednik's theorem in the special case $F = \mathbb{Q}$. Note that it is only in this case that the Shimura curve considered by Cherednik represents a moduli problem of abelian varieties. Furthermore, Drinfeld proved an "integral version" of this theorem which has the original version as a corollary. In his formulation appears the formal scheme $\widehat{\Omega}^2_{F_w}$ over Spec O_{F_w}, with "generic fiber" equal to $\Omega^2_{F_w}$, defined by Mumford, Deligne and Drinfeld. In particular, he interpreted the formal scheme $\widehat{\Omega}^2_{F_w}$, and its higher-dimensional versions $\widehat{\Omega}^n_{F_w}$ as formal moduli spaces of *special formal O_{B_w}-modules*, where B_w is the central division algebra over F_w with invariant $1/n$.

This integral uniformization theorem was generalized to higher-dimensional cases in [**26**]. In these cases, one uniformizes Shimura varieties associated to certain unitary groups over a totally real field F which at the archimedean places have signature $(1, n-1)$ at one place v, and signature $(0, n)$ at all others, and such that the associated CM-field K has two distinct places over the p-adic place w of F. (One has to be much, much more specific to force p-adic uniformization, cf. *loc. cit.*, p. 298–315.) Using these methods, Boutot and Zink [**2**] have given a conceptual proof of Cherednik's theorem for general totally real fields, and constructed at the same time integral models for the corresponding Shimura varieties. Such integral models were also constructed for general Shimura curves by Carayol [**4**]. In this context also falls the work of Varshavsky [**29, 30**], which concerns the p-adic uniformization of Shimura varieties associated to similar unitary groups, again where the p-adic place w splits in K (but not the construction of integral models).

In this paper, we give a new (very restricted) class of Shimura varieties which admit p-adic uniformization. For this class we prove p-adic uniformization for their generic fibers and, in certain cases, also p-adic uniformization for their integral models. The relevant reductive groups are defined in terms of two dimensional hermitian spaces for CM fields and the corresponding Shimura varieties represent a moduli problem of abelian varieties with additional structure. By extending the moduli problem integrally, we obtain integral models of these Shimura varieties which allow us to formulate and prove an "integral" version of our uniformization result. Behind this integral version of Theorem 1.1 is our interpretation of the Drinfeld formal halfplane $\widehat{\Omega}^2_F$, for a p-adic local field F and a quadratic extension K of F, as the formal moduli space of polarized two-dimensional O_K-modules of Picard type, established in a previous paper (1).

The simplest example is the following. Let K be an imaginary quadratic field, and let V be a hermitian vector space of dimension 2 over K of signature $(1,1)$. Let $G = \mathrm{GU}(V)$ be the group of unitary similitudes of V. For $C \subset G(\mathbb{A}^\infty)$ an open compact subgroup, there is a Shimura variety Sh_C with canonical model over $\mathbb{Q}$ whose complex points are given by

$$\mathrm{Sh}_C(\mathbb{C}) \simeq G(\mathbb{Q})\backslash[\mathbf{X} \times G(\mathbb{A}^\infty)/C],$$

where again $\mathbf{X} = \mathbb{C} \smallsetminus \mathbb{R}$, which is acted on by $G(\mathbb{R})$ *via* a fixed isomorphism $G_{\mathrm{ad}}(\mathbb{R}) \simeq \mathrm{PGL}_2(\mathbb{R})$.

Suppose that p is a prime that does *not* split in K and that $p \neq 2$ if p is ramified in K. Suppose that the local hermitian space $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is *anisotropic* and that C has the form $C = C^p \cdot C_p$, where C_p is the *unique* maximal compact subgroup of $G(\mathbb{Q}_p)$. Let $\bar{V}$ be the hermitian space over K which is positive definite, split at p, and locally coincides with V at all places $\neq \infty, p$, and let $I = \mathrm{GU}(\bar{V})$ be the corresponding group of unitary similitudes. Then there is an identification of the adjoint group $I_{\mathrm{ad}}(\mathbb{Q}_p)$ with $\mathrm{PGL}_2(\mathbb{Q}_p)$ and an action of $I(\mathbb{Q})$ on $G(\mathbb{A}^\infty)/C$.

Theorem 1.1. — *There is an isomorphism of algebraic curves over the completion of the maximal unramified extension $\breve{\mathbb{Q}}_p$ of $\mathbb{Q}_p$,*

$$\mathrm{Sh}_C \otimes_{\mathbb{Q}} \breve{\mathbb{Q}}_p \simeq \big(I(\mathbb{Q})\backslash[\Omega^2_{\mathbb{Q}_p} \times G(\mathbb{A}^\infty)/C]\big) \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$$

Next we give a simplified version of our main theorem about integral uniformization (2). Let K be a CM quadratic extension of a totally real field F of degree d over $\mathbb{Q}$, and let V be a hermitian vector space of dimension 2 over K with signature $(1,1)$ at every archimedean place of F. Let G be the group of unitary similitudes of V with multiplier in $\mathbb{Q}^\times$. For an open compact subgroup C of $G(\mathbb{A}^\infty)$, let Sh_C be

1. In [**16**], we excluded ramification in the case of even residue characteristic, and this restriction will thus be in force in the present paper.

2. Unexplained terms in the statement are defined in the main body of the text.