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REMARKS ON YU'S 'PROPERTY A' FOR DISCRETE METRIC SPACES AND GROUPS

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ABSTRACT. — Guoliang Yu has introduced a property on discrete metric spaces and groups, which is a weak form of amenability and which has important applications to the Novikov conjecture and the coarse Baum–Connes conjecture. The aim of the present paper is to prove that property in particular examples, like spaces with subexponential growth, amalgamated free products of discrete groups having property A and HNN extensions of discrete groups having property A.

RÉSUMÉ (*Remarques sur la propriété A de Yu pour les espaces métriques et les groupes discrets*)

Guoliang Yu a introduit une propriété sur les espaces métriques et les groupes discrets, qui est une forme faible de moyennabilité et qui a d'importantes applications à la conjecture de Novikov et la conjecture de Baum–Connes "coarse". Le but de cet article est de démontrer cette propriété dans des cas particuliers, tels que les espaces à croissance sous-exponentielle, les produits libres amalgamés de groupes discrets ayant la propriété A et les extensions HNN de groupes discrets ayant la propriété A.

1. Introduction

Let X be a discrete metric space. It is said to be of *bounded geometry* if there exists $N\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the number of elements in balls of given

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radius is uniformly bounded:

$$\forall x \in X, \quad \#B(x,R) \leq N(R). \tag{1.1}$$

In [19, Definition 2.1], Yu introduces a property on discrete metric spaces he calls property A, which is a weak form of amenability. It is shown in [10], [11], [19] that

- For every discrete group G with a left-invariant distance such that the resulting metric space has bounded geometry, G has property A if and only if it admits an amenable action on some compact space (or, equivalently, on its Stone-Čech compactification βG) [11, Theorem 3.3].
- With the same assumptions, if G has property A, then the Baum–Connes map for G is split injective [10, Theorem 3.2], hence G satisfies the Novikov Conjecture (see [3] for an introduction to the Baum–Connes conjecture and its relation to the Novikov conjecture). Moreover, the reduced group C^*-algebra $C_r^*(G)$ is exact, meaning that for every exact sequence of C^*-algebras

  $$0 \to J \longrightarrow A \longrightarrow A/J \to 0,$$

  the sequence obtained by taking spatial tensor products

  $$0 \to J \otimes_{\min} C_r^*(G) \longrightarrow A \otimes_{\min} C_r^*(G) \longrightarrow A/J \otimes_{\min} C_r^*(G) \to 0$$

  is exact (see [17] for a survey on exactness).
- Every discrete metric space with bounded geometry with property A satisfies the coarse Baum–Connes conjecture [19, Theorem 1.1] (see [13], [18] for an introduction to that conjecture).

That such impressive consequences result from that elementary property (see Definition 3.1) is quite remarkable. It was conjectured for a while that every discrete metric space has property A, but Gromov recently announced the construction of Cayley graphs that do not satisfy the property [7]. It remains important to determine classes of metric spaces or groups for which the property holds.

It is known that property A is true for amenable groups, semi-direct products of groups that have property A, asymptotically finite dimensional metric spaces with bounded geometry, hyperbolic groups in the sense of Gromov (see [8]). In this paper, it is proven that property A is true in each of the following cases, for a discrete metric space with bounded geometry X:

- $X \subset Y$, where Y is a metric space with property A;
- X has subexponential growth;
- $X = Y_1 \cup Y_2$, where (Y_1, Y_2) is an excisive pair;
- X is hyperbolic in the sense of Gromov;

- X is a group acting on a tree, such that the stabilizer of each vertex has property A. In particular, property A for groups is stable by taking amalgamated free products and HNN extensions.

We have tried in this paper to keep proofs as elementary and self-contained as possible, hoping to spark the interest of a broad range of readers.

2. Basic definitions

Let us recall a few elementary definitions from [13].

A metric space is said to be *proper* if every closed ball is compact.

Let X and Y be metric spaces. A (not necessarily continuous) map $f\colon X \to Y$ is said to be *proper* if the inverse image of any bounded set is bounded, and it is *coarse* if it is proper and if for every $R > 0$, there exists $S > 0$ such that for every x, $x' \in X$, $d(x, x') \leq R$ implies $d(f(x), f(x')) \leq S$.

Two coarse maps f, $g\colon X \to Y$ are *bornotopic* if there exists $R > 0$ such that $d(f(x), g(x)) \leq R$ for every $x \in X$. A coarse map $f\colon X \to Y$ is a *coarse equivalence* if there exists a coarse map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are bornotopic to the identity; X and Y are then said to be coarsely equivalent.

Two distances d and d' on X are *coarsely equivalent* if the identity $(X, d) \to (X, d')$ is a coarse equivalence.

A map $f\colon X \to Y$ is a *uniform embedding* if it induces a coarse equivalence between X and $f(X)$. This means that f is coarse, and that for every $R > 0$, there exists $S > 0$ such that $d(x, x') \geq S$ implies $d(f(x), f(x')) \geq R$ for all $x, x' \in X$.

LEMMA 2.1. — *Let G be a countable discrete group. Then up to coarse equivalence, there exists one and only one left-invariant distance on G for which the resulting metric space has bounded geometry.*

Proof. — Let e be the unit element in G. Let d and d' be such distances, and $\ell(g) = d(g, e)$, $\ell'(g) = d'(g, e)$ the associated length functions. Let $R > 0$. Since $\# B_d(e, R) < \infty$, there exists $S > 0$ such that for all $g \in B_d(e, R)$, $\ell'(g) \leq S$. By the left invariance, $\mathrm{Id}_G\colon (G, d) \to (G, d')$ is coarse. Similarly, $\mathrm{Id}_G\colon (G, d') \to (G, d)$ is coarse.

To prove the existence, let $f\colon G \to \mathbb{N}^*$ be a function such that $f^{-1}([0, n])$ is finite for every n, $f(g) = f(g^{-1})$ for all $g \in G$, and $f(g) = 0$ iff $g = 1$. Let

$$\ell(g) = \inf\bigl\{f(g_1) + \cdots + f(g_n)\colon g = g_1 \cdots g_n\bigr\}.$$

The distance $d(g, h) = \ell(g^{-1}h)$ is left-invariant and the resulting metric space has bounded geometry. □

If the group is finitely generated, one can take the distance associated to any finite system of generators. If G acts freely and co-compactly by isometries on a proper metric space X, and $x_0 \in X$ is arbitrary, then one can take $d(g,h) = \ell(g^{-1}h)$ where $\ell(g) = d(gx_0, x_0)$.

3. Property A, equivalent definitions

This section presents a few equivalent definitions of the property A introduced by Yu [19]. For a given metric space and $R > 0$, Δ_R will denote

$$\big\{(x,y) \in X \times X \colon d(x,y) \le R\big\}.$$

Definition 3.1. — (See [19, Definition 2.1].) A discrete metric space X is said to have property A if for any $R > 0$, $\varepsilon > 0$, there exist $S > 0$ and a family $(A_x)_{x\in X}$ of finite, nonempty subsets of $X \times \mathbb{N}$, such that

(i) $(y,n) \in A_x$ implies $(x,y) \in \Delta_S$;

(ii) for all $(x,y) \in \Delta_R$,

$$\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} \le \varepsilon.$$

Let us first recall the definition of a positive type kernel [12, Definition 5.1]. Let X be a set. A function $\varphi\colon X \times X \to \mathbb{R}$ is said to be a positive type kernel if $\varphi(x,y) = \varphi(y,x)$ for all x, $y \in X$, and if for every finitely supported, real-valued function $(\lambda_x)_{x\in X}$ on X, the following inequality holds:

$$\sum_{x,y\in X} \lambda_x \lambda_y \varphi(x,y) \ \ge\ 0. \tag{3.1}$$

A function $\varphi\colon X \times X \to \mathbb{R}$ is of positive type if and only if there exists a map $x \mapsto \eta_x$ from X to a real Hilbert space H such that $\varphi(x,y) = \langle \eta_x, \eta_y\rangle$ [12, Proposition 5.3].

Equivalent definitions listed in the proposition below clearly show that property A is a weak form of amenability. Indeed, (ii) and (iii) are Reiter's property (P1) and (P2) respectively, and (v) is Hulanicki's property [5].

Proposition 3.2. — *Let X be a discrete metric space with bounded geometry. The following are equivalent:*

(i) *X has property A;*

(ii) $\forall R > 0$, $\forall \varepsilon > 0$, $\exists S > 0$, $\exists (\xi_x)_{x\in X}$, $\xi_x \in \ell^1(X)$, $\mathrm{supp}(\xi_x) \subset B(x,S)$, $\|\xi_x\|_{\ell^1(X)} = 1$, *and* $\|\xi_x - \xi_y\|_{\ell^1(X)} \le \varepsilon$ *whenever* $d(x,y) \le R$;

(ii′) $\forall R > 0$, $\forall \varepsilon > 0$, $\exists S > 0$, $\exists (\chi_x)_{x\in X}$, $\chi_x \in \ell^1(X)$, $\mathrm{supp}(\chi_x) \subset B(x,S)$, $\|\chi_x - \chi_y\|_{\ell^1(X)}/\|\chi_x\|_{\ell^1(X)} \le \varepsilon$ *whenever* $d(x,y) \le R$;

(iii) $\forall R > 0$, $\forall \varepsilon > 0$, $\exists S > 0$, $\exists (\eta_x)_{x\in X}$, $\eta_x \in \ell^2(X)$, $\mathrm{supp}(\eta_x) \subset B(x,S)$, $\|\eta_x\|_{\ell^2(X)} = 1$, *and* $\|\eta_x - \eta_y\|_{\ell^2(X)} \le \varepsilon$ *whenever* $d(x,y) \le R$;

(iv) $\forall R>0$, $\forall \varepsilon>0$, $\exists S>0$, $\exists (\zeta_x)_{x\in X}$, $\zeta_x \in \ell^2(X\times\mathbb{N})$, $\operatorname{supp}(\zeta_x)\subset B(x,S)\times\mathbb{N}$, $\|\zeta_x\|_{\ell^2(X\times\mathbb{N})}=1$, *and* $\|\zeta_x-\zeta_y\|_{\ell^2(X\times\mathbb{N})}\le\varepsilon$ *whenever* $d(x,y)\le R$*;*

(v) $\forall R>0$, $\forall\varepsilon>0$, $\exists S>0$, $\exists\varphi\colon X\times X\to\mathbb{R}$ *of positive type such that* $\operatorname{supp}\varphi\subset\Delta_S$ *and* $|1-\varphi(x,y)|\le\varepsilon$ *whenever* $d(x,y)\le R$.

Proof. — (i) ⇔ (ii): noting that in (ii), ξ_x may be supposed to be nonnegative (since $\||\xi_x|-|\xi_y|\|_{\ell^1(X)}\le\|\xi_x-\xi_y\|_{\ell^1(X)}$), this is exactly [11, Lemma 3.5].

(ii) ⇒ (ii′): obvious.

(ii′) ⇒ (ii): let χ_x as in (ii′). Let $\xi_x=\chi_x/\|\chi_x\|_{\ell^1(X)}$. Then

$$\begin{aligned}\|\xi_x-\xi_y\|_1 &\le \frac{\|\chi_x-\chi_y\|_1}{\|\chi_x\|_1}+\|\chi_y\|_1\Big|\frac{1}{\|\chi_x\|_1}-\frac{1}{\|\chi_y\|_1}\Big|\\ &= \frac{\|\chi_x-\chi_y\|_1}{\|\chi_x\|_1}+\frac{|\|\chi_y\|_1-\|\chi_x\|_1|}{\|\chi_x\|_1}\le\frac{2\|\chi_x-\chi_y\|_1}{\|\chi_x\|_1}\,.\end{aligned}$$

(ii) ⇒ (iii): let ξ_x as in (ii). Define $\eta_x=|\xi_x|^{1/2}$. Then, denoting by $\int_X$ the summation on X, *i.e.* the integral with counting measure on X, one has

$$\begin{aligned}\|\eta_x-\eta_y\|^2_{\ell^2(X)} &= \int_X|\eta_x-\eta_y|^2\\ &\le \int_X|\eta_x^2-\eta_y^2| = \big\||\xi_x|-|\xi_y|\big\|_{\ell^1(X)}\le\|\xi_x-\xi_y\|_{\ell^1(X)}.\end{aligned}$$

(iii) ⇒ (ii): Let η_x as in (iii). We can suppose that $\eta_x\ge 0$. Let $\xi_x=\eta_x^2$. Then by the Cauchy-Schwarz inequality,

$$\begin{aligned}\|\xi_x-\xi_y\|_{\ell^1(X)} &= \int_X|\eta_x^2-\eta_y^2| = \int_X|\eta_x-\eta_y|(\eta_x+\eta_y)\\ &\le \|\eta_x-\eta_y\|_{\ell^2(X)}\|\eta_x+\eta_y\|_{\ell^2(X)}\le 2\|\eta_x-\eta_y\|_{\ell^2(X)}.\end{aligned}$$

(iii) ⇒ (iv): obvious.

(iv) ⇒ (iii): Let ζ_x as in (iv). Let $\eta_x(z)=\|\zeta_x(z,\cdot)\|_{\ell^2(\mathbb{N})}$. Then

$$\begin{aligned}\|\eta_x-\eta_y\|^2_{\ell^2(X)} &= \sum_{z\in X}\big|\|\zeta_x(z,\cdot)\|_{\ell^2(\mathbb{N})}-\|\zeta_y(z,\cdot)\|_{\ell^2(\mathbb{N})}\big|^2\\ &\le \sum_{z\in X}\big\|\zeta_x(z,\cdot)-\zeta_y(z,\cdot)\big\|^2_{\ell^2(X\times\mathbb{N})}=\|\zeta_x-\zeta_y\|^2_{\ell^2(X\times\mathbb{N})}.\end{aligned}$$

(iii) ⇒ (v): Let η_x as in (iii). Let $\varphi(x,y)=\langle\eta_x,\eta_y\rangle$. Then $\operatorname{supp}\varphi\subset\Delta_{2S}$ and if $d(x,y)\le R$, then $1-\varphi(x,y)=\frac12\|\eta_x-\eta_y\|^2_{\ell^2(X)}\le\frac12\varepsilon^2$.

(v) ⇒ (iii) is inspired from [4], proof of Theorem 13.8.6. The parallel would have been more apparent, had we introduced the concept of positive definite