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RANDOM WALKS IN $(\mathbb{Z}_+)^2$ WITH NON-ZERO DRIFT ABSORBED AT THE AXES

by Irina Kurkova & Kilian Raschel

Abstract. — Spatially homogeneous random walks in $(\mathbb{Z}_+)^2$ with non-zero jump probabilities at distance at most 1, with non-zero drift in the interior of the quadrant and absorbed when reaching the axes are studied. Absorption probabilities generating functions are obtained and the asymptotic of absorption probabilities along the axes is made explicit. The asymptotic of the Green functions is computed along all different infinite paths of states, in particular along those approaching the axes.

Résumé (*Marches aléatoires dans $\mathbb{Z}_+^2$ avec un drift non nul, absorbées au bord*)
Dans cet article, nous étudions les marches aléatoires du quart de plan ayant des sauts à distance au plus un, avec un drift non nul à l'intérieur et absorbées au bord. Nous obtenons de façon explicite les séries génératrices des probabilités d'absorption au bord, puis leur asymptotique lorsque le site d'absorption tend vers l'infini. Nous calculons également l'asymptotique des fonctions de Green le long de toutes les trajectoires, en particulier selon celles tangentes aux axes.

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1. Introduction

Random walks in angles of $\mathbb{Z}^d$ conditioned in the sense of Doob's h-transform never to reach the boundary nowadays arouse enough interest in the mathematical community as they appear in several distinct domains.

An important class of such walks is the so-called "non-colliding" random walks. These walks are the processes $(Z_1(n), \ldots, Z_k(n))_{n\geq 0}$ composed of k independent and identically distributed random walks that never leave the Weyl chamber $W = \{z \in \mathbb{R}^k : z_1 < \cdots < z_k\}$. The distances between these random walks $U(n) = (Z_2(n) - Z_1(n), \ldots, Z_k(n) - Z_{k-1}(n))$ give a $k-1$ dimensional random process whose components are positive. These processes appear in the eigenvalue description of important matrix-valued stochastic processes: see [7] for an old well-known result on the eigenvalues of the process version of the Gaussian Unitary Ensemble and e.g. [5], [20], [19], [11], [13]. They are found in the analysis of corner-growth model, see [17] and [18]. Moreover, interesting connections between non-colliding walks, random matrices and queues in tandem are the subject of [28]. Paper [8] reveals a rather general mechanism of the construction of the suitable h-transform for such processes. But processes whose components are distances between independent random walks are not the only class of interest. In [21], random walks with exchangeable increments and conditioned never to exit the Weyl chamber are considered. In [29], the authors study a certain class of random walks, namely $(X_i(n))_{1\leq i\leq k} = (|\{1 \leq m \leq n : \xi_m = i\}|)_{1\leq i\leq k}$, where $(\xi_m, m \geq 1)$ is a sequence of i.i.d. random variables with common distribution on $\{1, 2, \ldots, k\}$. The authors identify in law their conditional version with a certain path-transformation. In [26] and [27], O'Connell relates these objects to the Robinson-Schensted algorithm.

Another important area where random processes in angles of $\mathbb{Z}^d$ conditioned never to reach the boundary appear is "quantum random walks". In [2], Biane constructs a quantum Markov chain on the von Neumann algebra of $\mathrm{SU}(n)$ and interprets the restriction of this quantum Markov chain to the algebra of a maximal torus of $\mathrm{SU}(n)$ as a random walk on the lattice of integral forms on $\mathrm{SU}(n)$ with respect to this maximal torus. He proves that the restriction of the quantum Markov chain to the center of the von Neumann algebra is a Markov chain on the same lattice obtained from the preceding by conditioning it in Doob's sense to exit a Weyl chamber at infinity. In [3], Biane extends these results to the case of general semi-simple connected and simply connected compact Lie groups, the basic notion being that of the minuscule weight. The corresponding random walk on the weight lattice in the interior of the Weyl chamber can be obtained as follows: if $2l$ is the order of the associated Weyl group, one draws the vector corresponding to the minuscule weight and its $l-1$

conjugates under the Weyl group ; then one translates these vectors to each point of the weight lattice in the interior of the Weyl chamber and assigns to them equal probabilities of jumps $1/l$.

For example, in the case U(3), the Weyl chamber of the corresponding Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is the "angle $\pi/3$", that is to say the domain of $(\mathbb{R}_+)^2$ delimited on the one hand by the x-axis and on the other by the axis making an angle equal to $\pi/3$ with the x-axis. One gets a spatially homogeneous random walk in the interior of the weights lattice, as in the left-hand side of Picture 1, the arrows designing transition probabilities equal to $1/3$. In the cases of the Lie algebras $\mathfrak{sp}_4(\mathbb{C})$ or $\mathfrak{so}_5(\mathbb{C})$, the Weyl chamber is the angle $\pi/4$, see the second picture of Figure 1 for the transition probabilities. Both of these random walks can be of course thought as walks in $(\mathbb{Z}_+)^2$ with transition probabilities drawn in the third and fourth pictures of Figure 1. Biane shows that the

FIGURE 1. The walks on weights lattice of classical algebras—above, $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$-–can be viewed as random walks on ${\mathbb{Z}_+}^d$

suitable Doob's h-transform $h(x, y)$ for these random walks is the dimension of the representation with highest weight $(x-1, y-1)$. In [1], again thanks to algebraic methods, he computes the asymptotic of the Green functions $G_{x,y}$ for the random walk with jump probabilities $1/3$ in the angle $\pi/3$ on the Picture 1, absorbed at the boundary, $x, y \to \infty$ and $y/x \to \tan(\gamma)$, γ lying in $\in [\epsilon, \pi/2-\epsilon]$, $\epsilon > 0$. The asymptotic of the Green functions as $y/x \to 0$ or $y/x \to \infty$ could not be found by these technics.

In [3] Biane also studies some extensions to random walks with drift: these are spatially homogeneous random walks in the same Weyl chambers, with the same non-zero jump probabilities as previously, but now these jump probabilities are admitted not to be all equal to $1/l$, so that the mean drift vector may have positive coordinates. Due to Choquet-Denis theory, in [3], he finds all minimal non-negative harmonic functions for these random walks. Nevertheless this approach seems not allow to find the Martin compactification of these random walks, nor to compute the asymptotic of the Green functions along different paths.

In [14], Ignatiouk-Robert obtains, under general assumptions and for all $d \geq 2$, the Martin boundary of some random walks in the half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$ killed on the boundary. In this paper and in [15], Ignatiouk-Robert proposes a new large deviation approach to the analysis of the Martin boundary combined with the ratio-limit theorem for Markov-additive processes. Ignatiouk-Robert and Loree develop this original approach in a recent paper [16] and apply it with success to the analysis of spatially homogeneous random walks in $(\mathbb{Z}_+)^2$ killed at the axes, under hypotheses of unbounded jump probabilities (more precisely, having exponential decay) and non-zero drift. They compute the Martin compactification for these random walks and therefore obtain the full Martin boundary. These methods seem not to be powerful for a more detailed study, as for the computation of the asymptotic of the Green functions, or for the computation of the absorption probabilities at different points on the axes, or for the enumeration of lattice walks (see [4] and references therein for the study of this last problem for lattice walks on $(\mathbb{Z}_+)^2$ by analytic methods).

They also seem to be difficult to generalize to the random walks with zero drift.

In this paper we would like to study in detail the spatially homogeneous random walks $(X(n), Y(n))_{n\geq 0}$ in $(\mathbb{Z}_+)^2$ with jumps at distance at most 1. We denote by $\mathbb{P}(X(n+1) = i_0 + i, Y(n+1) = j_0 + j \mid X(n) = i_0, Y(n) = j_0) = p_{(i_0,j_0),(i+i_0,j+j_0)}$ the transition probabilities and do the hypothesis:

(H1) *For all* (i_0, j_0) *such that* $i_0 > 0, j_0 > 0$, $p_{(i_0,j_0),(i_0,j_0)+(i,j)}$ *does not depend on* (i_0, j_0) *and can thus be denoted by* p_{ij}.

(H2) $p_{ij} = 0$ *if* $|i| > 1$ *or* $|j| > 1$.

(H3) *The boundary* $\{(0, 0)\} \cup \{(i, 0) : i \geq 1\} \cup \{(0, j) : j \geq 1\}$ *is absorbing.*

(H4) *In the list* $p_{11}, p_{10}, p_{1-1}, p_{0-1}, p_{-1-1}, p_{-1,0}, p_{-11}, p_{0-1}$ *there are no three consecutive zeros.*

The last hypothesis (H4) is purely technical and avoids studying degenerated random walks.

In a companion paper we gave a rather complete analysis of such random walks under a simplifying hypothesis that also

(H2′) $p_{-11} = p_{11} = p_{1-1} = p_{-1-1} = 0$.

This hypothesis made the analysis more transparent for several reasons. First, the problem to find the generating functions of absorption probabilities, defined in (1) and (2) below, could be reduced to the resolution of Riemann boundary value problems on contours inside unit discs, where these functions are holomorphic, being generating functions of probabilities. These contours under general hypothesis (H2) may lie outside the unit disc, so that we are obliged first to continue these functions as holomorphic, then to exploit this continuation. Secondly, the conformal gluing function responsible for the conversion