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## ON THE IWAHORI WEYL GROUP

BY TIMO RICHARZ

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ABSTRACT. — Let  $F$  be a discretely valued complete field with valuation ring  $\mathcal{O}_F$  and perfect residue field  $k$  of cohomological dimension  $\leq 1$ . In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected  $F$ -groups to the case of arbitrary connected reductive  $F$ -groups. If  $k$  is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4].

Let  $F$  be a discretely valued complete field with valuation ring  $\mathcal{O}_F$  and perfect residue field  $k$  of cohomological dimension  $\leq 1$ . In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected  $F$ -groups to the case of arbitrary connected reductive  $F$ -groups. If  $k$  is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4]. Furthermore, we study the length function on the Iwahori-Weyl group, and use it to determine the number of points in a Bruhat cell, when  $k$  is a finite field. Except for Lemma 1.3 below, the results are independent of [4], and are directly based on the work of Bruhat and Tits [2], [3].

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Let  $\bar{F}$  be the completion of a separable closure of  $F$ . Let  $\check{F}$  be the completion of the maximal unramified subextension with valuation ring  $\mathcal{O}_{\check{F}}$  and residue field  $\bar{k}$ . Let  $I = \text{Gal}(\bar{F}/\check{F})$  be the inertia group of  $\check{F}$ , and let  $\Sigma = \text{Gal}(\check{F}/F)$ .

Let  $G$  be a connected reductive group over  $F$ , and denote by  $\mathcal{B} = \mathcal{B}(G, F)$  the enlarged Bruhat-Tits building. Fix a maximal  $F$ -split torus  $A$ . Let  $\mathcal{A} = \mathcal{A}(G, A, F)$  be the apartment of  $\mathcal{B}$  corresponding to  $A$ .

**1.1. Definition of the Iwahori-Weyl group.** — Let  $M = Z_G(A)$  be the centralizer of  $A$ , an anisotropic group, and let  $N = N_G(A)$  be the normalizer of  $A$ . Denote by  $W_0 = N(F)/M(F)$  the relative Weyl group.

DEFINITION 1.1. — i) The *Iwahori-Weyl group*  $W = W(G, A, F)$  is the group

$$W \stackrel{\text{def}}{=} N(F)/M_1,$$

where  $M_1$  is the unique parahoric subgroup of  $M(F)$ .

ii) Let  $\mathfrak{a} \subset \mathcal{A}$  be a facet and  $P_{\mathfrak{a}}$  the associated parahoric subgroup. The subgroup  $W_{\mathfrak{a}}$  of the Iwahori-Weyl group corresponding to  $\mathfrak{a}$  is the group

$$W_{\mathfrak{a}} \stackrel{\text{def}}{=} P_{\mathfrak{a}} \cap N(F)/M_1.$$

The group  $N(F)$  operates on  $\mathcal{A}$  by affine transformations

$$(1.1) \quad \nu : N(F) \longrightarrow \text{Aff}(\mathcal{A}).$$

The kernel  $\ker(\nu)$  is the unique maximal compact subgroup of  $M(F)$  and contains the compact group  $M_1$ . Hence, the morphism (1.1) induces an action of  $W$  on  $\mathcal{A}$ .

Let  $G_1$  be the subgroup of  $G(F)$  generated by all parahoric subgroups, and define  $N_1 = G_1 \cap N(F)$ . Fix an alcove  $\mathfrak{a}_C \subset \mathcal{A}$ , and denote by  $B$  the associated Iwahori subgroup. Let  $\mathbb{S}$  be the set of simple reflections at the walls of  $\mathfrak{a}_C$ . By Bruhat and Tits [3, Prop. 5.2.12], the quadruple

$$(1.2) \quad (G_1, B, N_1, \mathbb{S})$$

is a (double) Tits system with affine Weyl group  $W_{\text{af}} = N_1/N_1 \cap B$ , and the inclusion  $G_1 \subset G(K)$  is  $B$ - $N$ -adapted of connected type.

LEMMA 1.2. — i) *There is an equality  $N_1 \cap B = M_1$ .*

ii) *The inclusion  $N(F) \subset G(F)$  induces a group isomorphism  $N(F)/N_1 \xrightarrow{\cong} G(F)/G_1$ .*

*Proof.* — The group  $N_1 \cap B$  operates trivially on  $\mathcal{A}$  and so is contained in  $\ker(\nu) \subset M(F)$ . In particular,  $N_1 \cap B = M(F) \cap B$ . But  $M(F) \cap B$  is a parahoric subgroup of  $M(F)$  and therefore equal to  $M_1$ .

The group morphism  $N(F)/N_1 \rightarrow G(F)/G_1$  is injective by definition. We have to show that  $G(F) = N(F) \cdot G_1$ . This follows from the fact that the inclusion  $G_1 \subset G(F)$  is  $B$ - $N$ -adapted, cf. [2, 4.1.2].  $\square$

Kottwitz defines in [5, §7] a surjective group morphism

$$(1.3) \quad \kappa_G : G(F) \longrightarrow X^*(Z(\hat{G})^I)^\Sigma.$$

Note that in [*loc. cit.*] the residue field  $k$  is assumed to be finite, but the arguments extend to the general case.

LEMMA 1.3. — *There is an equality  $G_1 = \ker(\kappa_G)$  as subgroups of  $G(F)$ .*

*Proof.* — For any facet  $\mathfrak{a}$ , let  $\text{Fix}(\mathfrak{a})$  be the subgroup of  $G(F)$  which fixes  $\mathfrak{a}$  pointwise. The intersection  $\text{Fix}(\mathfrak{a}) \cap \ker(\kappa_G)$  is the parahoric subgroup associated to  $\mathfrak{a}$ , cf. [4, Proposition 3]. This implies  $G_1 \subset \ker(\kappa_G)$ . For any facet  $\mathfrak{a}$ , let  $\text{Stab}(\mathfrak{a})$  be the subgroup of  $G(F)$  which stabilizes  $\mathfrak{a}$ . Fix an alcove  $\mathfrak{a}_C$ . There is an equality

$$(1.4) \quad \text{Fix}(\mathfrak{a}_C) \cap G_1 = \text{Stab}(\mathfrak{a}_C) \cap G_1,$$

and (1.4) holds with  $G_1$  replaced by  $\ker(\kappa_G)$ , cf. [4, Lemma 17]. Assume that the inclusion  $G_1 \subset \ker(\kappa_G)$  is strict, and let  $\tau \in \ker(\kappa_G) \setminus G_1$ . By Lemma 1.2 ii), there exists  $g \in G_1$  such that  $\tau' = \tau \cdot g$  stabilizes  $\mathfrak{a}_C$ , and hence  $\tau'$  is an element of the Iwahori subgroup  $\text{Stab}(\mathfrak{a}_C) \cap \ker(\kappa_G)$ . This is a contradiction, and proves the lemma.  $\square$

By Lemma 1.2, there is an exact sequence

$$(1.5) \quad 1 \longrightarrow W_{\text{af}} \longrightarrow W \xrightarrow{\kappa_G} X^*(Z(\hat{G})^I)^\Sigma \longrightarrow 1.$$

The stabilizer of the alcove  $\mathfrak{a}_C$  in  $W$  maps isomorphically onto  $X^*(Z(\hat{G})^I)^\Sigma$  and presents  $W$  as a semidirect product

$$(1.6) \quad W = X^*(Z(\hat{G})^I)^\Sigma \ltimes W_{\text{af}}.$$

For a facet  $\mathfrak{a}$  contained in the closure of  $\mathfrak{a}_C$ , the group  $W_{\mathfrak{a}}$  is the parabolic subgroup of  $W_{\text{af}}$  generated by the reflections at the walls of  $\mathfrak{a}_C$  which contain  $\mathfrak{a}$ .

THEOREM 1.4. — *Let  $\mathfrak{a}$  (resp.  $\mathfrak{a}'$ ) be a facet contained in the closure of  $\mathfrak{a}_C$ , and let  $P_{\mathfrak{a}}$  (resp.  $P_{\mathfrak{a}'}$ ) be the associated parahoric subgroup. There is a bijection*

$$\begin{aligned} W_{\mathfrak{a}} \backslash W / W_{\mathfrak{a}'} &\xrightarrow{\cong} P_{\mathfrak{a}} \backslash G(F) / P_{\mathfrak{a}'} \\ W_{\mathfrak{a}} w W_{\mathfrak{a}'} &\longmapsto P_{\mathfrak{a}} n_w P_{\mathfrak{a}'}, \end{aligned}$$

where  $n_w$  denotes a representative of  $w$  in  $N(F)$ .