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DIFFERENTIAL EQUATIONS AND ALGEBRAIC TRANSCENDENTS: FRENCH EFFORTS AT THE CREATION OF A GALOIS THEORY OF DIFFERENTIAL EQUATIONS 1880–1910

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Abstract. — A "Galois theory" of differential equations was first proposed by Émile Picard in 1883. Picard, then a young mathematician in the course of making his name, sought an analogue to Galois's theory of polynomial equations for linear differential equations with rational coefficients. His main results were limited by unnecessary hypotheses, as was shown in 1892 by his student Ernest Vessiot, who both improved Picard's results and altered his approach, leading Picard to assert that his lay closest to the path of Galois. The subject became interesting to a number of French researchers in the next decade and more, most importantly Jules Drach, whose flawed 1898 doctoral thesis led to a further reworking of the subject by Vessiot. The present paper recounts these events, looking at the tools created and at the interpretation of the Galois legacy manifest in these different attempts.

Résumé (Équations différentielles et transcendants algébriques : les efforts français sur la création d'une théorie de Galois pour les équations différentielles 1880–1910)

Une « théorie de Galois » pour les équations différentielles a été créée pour la première fois par Émile Picard en 1883. Picard, à cette époque un jeune mathématicien qui cherchait faire une réputation, a façonné une théorie analogue à celle des équations algébriques de Galois pour les équations différentielles linéaires à coefficients rationnels. Ses résultats étaient limités par des hypothèses superflues, un fait démontré en 1892 par son élève Ernest Vessiot, qui a amélioré les résultats de Picard en modifiant son approche.

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Cette modification a mené Picard à affirmer que c'était son approche à lui qui restait plus fidèle au chemin tracé par Galois. Le sujet a intéressé plusieurs chercheurs en France dans les années qui suivirent, le plus important étant Jules Drach, dont la thèse erronée de 1898 a provoqué encore une intervention de Vessiot. Cet article relate ces évènements, en considérant les outils utilisés et l'interprétation du legs de Galois manifestée dans une série d'efforts divers.

1. INTRODUCTION

The reception of the work of Évariste Galois on the solution of polynomial equations, and the ways in which the name of Galois became emblematic for a certain kind of mathematical creativity, make a complicated story. In this paper we take on the question of what it meant in the context of the study of differential equations. As the pervasive presence of groups in mathematics dawned on at least some important researchers—Felix Klein, Sophus Lie, Henri Poincaré—the idea of employing an analogous theory for differential equations was enunciated by Poincaré's associate Émile Picard, whose first publication on the subject was in 1883. This was followed by further work of Picard, Ernest Vessiot, Jules Drach, and other French mathematicians, leading on the one hand to what has come to be called the Picard-Vessiot theory, an object of renewed research interest in recent years [Magid 1999]; and the "logical" integration theory of Drach. All three of these writers claimed their own approach as being the true heir to the essential ideas of Galois. In what follows we try to unpack what they meant by this, why there was some divergence, and what the claim means about values in mathematics and the relations between algebra and analysis in the late nineteenth century.

It was to become a commonplace of twentieth-century mathematics to pattern one theory on another, attempting to find analogous components and then exploiting similarities of "structure" in order to find results. Indeed, the structural turn has been described by Corry and others as characteristic of much pure mathematics of the twentieth century, though the idea of analogical building of theories is only one component of this approach [Corry 2004]. In fact, the notion of a mathematical theory was in transition in the last decades of the nineteenth century, when the term was used commonly in a non-technical way to denote a body of connected results on a single subject. Formal theories in the sense of Russell and others were a construction that was to come in the future. Indicative of the way in which the term was used are the following remarks of Francesco Brioschi, a senior observer describing what he sees as a modern tendency:

> The characteristic note of modern progress in mathematical studies can be recognized in the contribution that each special theory—that of functions, of substitutions, of forms, of transcendents, geometrical theories and so on—brings to the study of problems for which in other times only one seemed necessary ... France, which, following the disaster of 1870, drew from it new and powerful scientific vitality, and has given proof of it in every realm of knowledge, has not remained outside this movement...[1] [Brioschi 1889, 72].

Despite the superficial resemblance between the problem of solving a polynomial equation and that of solving an ordinary or partial differential equation, the idea of creating a Galois theory for differential equations faced formidable obstacles. In the case of the original Galois theory, one starts with a polynomial equation. The theory relies on the correspondence between a splitting field that is an algebraic extension of $\mathbb{Q}$ and the automorphism group of the polynomial, that is, a subgroup of the permutation group of the roots. The main theorem of the subject states that if that group is solvable then the equation is solvable by radicals; this requires the notion of normal subgroup, a key construct of the theory. Yet the words with which we describe these objects easily now all emerged with their present meaning during and after the period we are discussing. In particular, the relationship between substitution groups (in the sense of Camille Jordan's 1870 treatise) and what Sophus Lie termed "transformation groups" was seen by many writers (including Jordan and Lie) as one of analogy rather than of identity of structure; and fundamental features of today's group concept (notably the presence of inverses) had a problematic status. Similarly the notion of an entity called a field, while adumbrated for example in Dedekind's work, existed alongside the notion of a slightly more fluid concept issuing from the work of Kronecker, the *domain of rationality*. The shifting meanings of these terms and a resulting

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1 "La nota caratteristica del progresso moderno degli studi matematici deve rintracciarsi nel contributo che ciascuna speciale teoria, quella delle funzioni, delle sostituzioni, delle forme, dei transcendenti, le geometriche e così via, porta nello studio di problemi pel quale in altri tempi forse una sola fra esse sembrava necessaria ... La Francia, la quale dalle sciagure del 1870 seppe ritirarre nuova e potente vitalità scientifica, e ne ha dato ampie prove in ogni ramo dello scibile, non rimase estranea a quel movimento ...".

vagueness in understanding the relationship between them pervades the work that we shall discuss in what follows.[2]

Thus when we move to the context of differential equations, we are immediately faced with a mass of difficulties. The "obvious" corresponding object to the rationals is the field of rational functions (in one variable) but since there is no result corresponding to the fundamental theorem of algebra there is not an evident analogy to the splitting field. Other complications include the fact that in the case of a differential equation of order p the set of solutions, far from finite, depends on up to p continuous parameters, and hence the groups involved would be continuous groups. This is where Lie's theory comes in: the analogy to the symmetric group of finite permutations of roots is, in Lie's work, the general linear group, and the appropriate subgroups are those that leave the equation fixed while transforming the solutions into each other (in which case the equation is usually described as *admitting* the transformation). Nothing here is as simple as in the algebraic context, and the search for the appropriate analogous structures was a large part of the struggle faced by the researcher.

Despite all this, in the years before 1880 many researchers had identified specific points of analogy between the theory of polynomial equations and those of linear differential equations, and this gave reason for optimism. Euler's complete solution of homogeneous linear equations with constant coefficients through the very mechanism of looking at a closely analogous polynomial equation dated from 1750 [Euler 1753]. Euler begins with a linear ordinary differential equation of order n. Then "ante omnia ex ea formetur haec expressio Algebraica $P = A + Bz + Cz^2 + Dz^3 + Ez^4 +$etc. cuius quaeratur omnes factores simplices..."[3]. Here the algebraic character of the analogy is made explicit—the expression P is repeatedly referred to as an algebraic formula, with the word algebraic capitalized. The correspondence is established between the *order* of the differential equation and the *degree* of the associated polynomial employed as a tool in its solution. (Euler in fact uses the same term, *gradus,* for both.)

By 1881, Paul Appell took up the question of the analogy in a context remarkably close to that of Galois, seeking differential analogies to symmetric functions of the roots. In a two-page introduction, Appell gave an

2 It is worth noting that the French term "structure" was explicitly introduced in a closely related context by one of the principal actors we discuss, Ernest Vessiot (1865-1952), who used this word as a translation for Sophus Lie's *Zusammensetzung.* [Hawkins 2000, 168], [Vessiot & de Tannenberg 1889, 137].

3 "... before anything else one forms from it this algebraic expression ... of which all the simple factors are sought."

extended survey of earlier work that had explored other features of the analogy between the two mathematical contexts [Appell 1881, 391-392], aspects of which were very well known in the Paris mathematical world of that time. Appell mentions first Lagrange's result that a reduction of order of a linear differential equation is possible when a solution is known, analogous to using a linear factor to reduce the degree of a polynomial equation. He then rapidly lists work of Liouville, Libri, Frobenius and others, noting in particular the idea of irreducibility of a differential equation due to Frobenius. Much closer to home he notes recent work of Jules Tannery (from 1874) expounding and extending the work of Lazarus Fuchs on linear differential equations; and the 1879 thesis of Gustave Floquet exploiting an analogy with polynomials through the use of factorization of a differential equation. These works made familiar the notion of a fundamental system of n (linearly independent) solutions of a linear differential equation of order n, an idea due to Fuchs, and demonstrated some of its utility.[4] The same year, 1879, saw two papers by E. Laguerre who discussed the question of invariance of a linear differential equation under a transformation of the variables [Laguerre 1879a], [Laguerre 1879b]. Thus Appell (and Picard, soon to explore this path) entered into the study of the subject at a time when such analogies were being actively and widely explored both in France and elsewhere.

If $y_1, \dots, y_n$ is a fundamental system of solutions Appell's own work identified the analogue of the symmetric functions of the roots:

> the functions in question are polynomials in $y_1, \dots, y_n$ and their derivatives which reappear multiplied by a non-zero constant when we replace $y_1, \dots, y_n$ by the elements $z_1, \dots, z_N$ of another fundamental system, that is when we make a linear substitution of the form $y_i = C_{i1}z_1 + C_{i2}z_2 + \cdots + C_{in}z_n \dots$ [5] [Appell 1881, 392]

We see here the analogue of algebraic invariants in the presence of invariance up to a constant multiple. Appell's key theorem states that every such function for a monic linear differential equation of order n may be expressed as an algebraic function of the coefficients multiplied by $e^{-\int a_1 dx}$ where a_1 is the coefficient of the term of degree $n-1$. Without going into

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4 Jordan used this idea around the same time, in [Jordan 1873/74], on a suggestion by Hamburger. I thank F. Brechenmacher for this information.

5 "Les fonctions en question sont des fonctions algébriques entières de $y_1, \dots, y_n$ et de leurs dérivées qui se reproduisent multipliées par un facteur constant différent de zéro quand on remplace $y_1, \dots, y_n$ par les éléments $z_1, \dots, z_N$ d'un autre système fondamental, c'est-à-dire quand on fait une substitution linéaire de la forme $y_i = C_{i1}z_1 + C_{i2}z_2 + \cdots + C_{in}z_n \dots$"