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COLEMAN'S $\mathscr{L}$-INVARIANT AND FAMILIES OF MODULAR FORMS

by

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Abstract. — We prove the conjecture of Mazur, Tate, and Teitelbaum with Coleman's $\mathscr{L}$-invariant for a newform f of arbitrary weight $k_0 \geq 2$ of split multiplicative type at a prime $p > 2$. The key step in the proof is to show that Coleman's $\mathscr{L}$-invariant is given by $\mathscr{L}(f) = -2p^{k_0/2}\alpha'(k_0)$, where $\alpha(k)$ is the eigenvalue of U_p acting on the germ of a Coleman family f_k passing through f at $k = k_0$.

Résumé **(Représentations ℓ-adiques de groupes p-adiques).** — On démontre une conjecture de Mazur, Tate et Teitelbaum, en termes de l'invariant $\mathscr{L}$ de Coleman, pour une forme primitive f de poids arbitraire $k_0 \geq 2$ et de type multiplicatif déployé en un nombre premier $p > 2$. Le point clé de la preuve consiste à montrer que l'invariant $\mathscr{L}$ de Coleman est donné par $\mathscr{L}(f) = -2p^{k_0/2}\alpha'(k_0)$, où $\alpha(k)$ est la valeur propre de U_p agissant sur le germe d'une famille de Coleman f_k passant par f en $k = k_0$.

Statement of results

Let p be a prime > 2 and N be a positive integer with $p \nmid N$. Let f be a classical newform over $\Gamma_0(Np)$ of even weight $k_0 + 2 \geq 2$ and assume f is split multiplicative at p, thus

$$a_p(f) = p^{k_0/2}$$

where $a_p(f)$ is the eigenvalue of the U-operator at p acting on f. Under these hypotheses, Coleman [**2**] defined an $\mathscr{L}$-invariant $\mathscr{L}(f)$ which he conjectured to be equal to the higher weight Mazur-Tate-Teitelbaum $\mathscr{L}$-invariant [**16**]. In this paper we will prove Coleman's conjecture. More precisely, let $\mathscr{X} := \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ with $\mathbb{Z}$ embedded in $\mathscr{X}$ diagonally and let $L_p(f, -) : \mathscr{X} \longrightarrow \mathbb{C}_p$ be the p-adic L-function attached to f as in [**16**]. We will prove the following theorem.

Main Theorem. — $L_p'(f, 1 + k_0/2) = \mathscr{L}(f) \cdot L_\infty(f, 1 + k_0/2)$.

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In the special case of weight two ($k_0 = 0$), in which case, $\mathscr{L}(f)$ takes the familiar form $\mathscr{L}(f) = \log(q_f)/\mathrm{ord}(q_f)$ when f has rational Fourier coefficients, this was conjectured by Mazur, Tate, and Teitelbaum [**16**] and proved by Ralph Greenberg and the author in [**11, 12**]. In case f is split multiplicative of weight $k_0 + 2 > 2$, Mazur, Tate, and Teitelbaum offered no precise formula for $\mathscr{L}(f)$, but they did predict that $\mathscr{L}(f)$ could be described purely in terms of local p-adic data associated to f and that, in particular, $\mathscr{L}(f)$ should not change when f is twisted by a Dirichlet character χ with $\chi(p) = 1$. Three separate and apparently independent definitions of $\mathscr{L}(f)$ were later proposed. The first was given by Jeremy Teitelbaum [**18**], but only in the case where f corresponds to a quaternionic modular form via the Jacquet-Langands correspondence. Robert Coleman gave an analogous definition in [**2**] in the general case, which we will briefly recall in section 2 of this paper. A third definition was proposed by Fontaine and Mazur [**15**] based on Fontaine's theory of semistable p-adic galois representations. These three definitions gave rise to three separate conjectures of Mazur-Tate-Teitelbaum type. All three of these conjectures have now been proved.

The $\mathscr{L}$-invariants of Coleman and Teitelbaum can be approximated p-adically on a computer, which enabled early numerical confirmation of the Coleman and Teitelbaum conjectures [**6, 7, 18**]. On the other hand, the Fontaine-Mazur $\mathscr{L}$-invariant appears to be beyond the reach of a computer. Nevertheless, it was the Fontaine-Mazur version of the conjecture that was the first to be proved – in 1996, by Kato, Kurihara, and Tsuji [**13**]. The Coleman version of the conjecture was established by the author shortly thereafter and described in [**17**], thus also proving indirectly that the Fontaine-Mazur and Coleman $\mathscr{L}$-invariants are the same. Coleman and Iovita [**5**] later gave a direct proof that *all three* $\mathscr{L}$-invariants—including Teitelbaum's invariant when it is defined—are the same. For an excellent overview of the history of the $\mathscr{L}$-invariant and the Mazur-Tate-Teitelbaum conjecture, see Colmez's survey [**9**]. The connection with Kato's Euler systems and the p-adic Birch-Swinnerton-Dyer conjecture, including the proof by Kato, Kurihara, and Tsuji given in the language of (φ, Γ)-modules, is also beautifully described in Colmez's Bourbaki seminar notes [**8**].

As in the weight two case (see [**11, 12**]), our proof of Coleman's conjecture in the higher weight case divides naturally into two steps (Theorems A and B below). To state Theorems A and B, we first recall that Coleman [**4**] constructed a p-adic analytic family f_k of overconvergent p-adic modular forms passing through our fixed newform f. This family is defined for k in an open set $B \subseteq \mathscr{X}$ containing k_0 and satisfies $f_{k_0} = f$. Coleman's family is an eigenfamily for the U-operator and we may therefore consider the eigenvalue $\alpha(k)$ of U acting on f_k. The function $\alpha(k)$ is a p-adic analytic function of $k \in B$ so we may consider the derivative of α at the special point $k_0 \in B$.

Theorem A. — $L_p'(f, 1 + k_0/2) = -2 \cdot p^{-k_0/2} \cdot \alpha'(k_0) \cdot L_\infty(f, 1 + k_0/2)$.

Just as in the weight two case, the proof of Theorem A depends on the existence of a two variable p-adic L-function with certain properties. The existence of such a p-adic L-function was proved in the higher weight case in [**17**]. With the two-variable

p-adic L-function in hand, the proof of theorem A proceeds exactly as in the weight two case (see [**11, 12**]).

The rest of this note is dedicated to proving the following theorem.

Theorem B. — $\mathscr{L}(f) = -2 \cdot p^{-k_0/2} \cdot \alpha'(k_0)$.

The Main Theorem is an immediate consequence of Theorems A and B. We remark that Colmez [**10**] has also proven Theorem B, but in terms of the Fontaine-Mazur $\mathscr{L}$-invariant.

1. The Gauss-Manin connection with Frobenius structure

We adopt Coleman's notations as in [**2**] with only one modification. Namely, we will add full level 2 structure to the moduli space. This rigidifies the setup and simplifies the calculations (see especially the proof of Proposition 3.1(2)). We let X be the modular curve $X(Np, 2)$ with level Np structure (a cyclic subgroup of order Np) plus full level 2 structure. (If $2|N$ we assume that the additional level 2 structure extends the 2-part of the level N structure.) The p-adic rigid analytic space X^{an} attached to X is the union of three disjoint parts, namely,

$$X^{an} = Z_\infty \cup W \cup Z_0$$

where Z_∞ and Z_0 are the ordinary affinoids containing the ∞ and 0-cusps respectively, and W is the union of the supersingular annuli. Following Coleman, we write $W_\infty = Z_\infty \cup W$ and $W_0 = Z_0 \cup W$.

Let $Y = Y(Np, 2)$ denote X with the cusps deleted. Let $\pi : E \longrightarrow Y$ be the universal elliptic curve with level structure over Y and let $\mathscr{H}$ be the relative de Rham cohomology sheaf over X with log singularities at the cusps. Then $\mathscr{H}$ is a coherent $\mathscr{O}$-module locally free of rank 2 over X. As Katz explains in [**14**] we have a canonical decomposition

$$\mathscr{H} = \underline{\omega}^{-1} \oplus \underline{\omega}$$

where $\underline{\omega} := \pi_* \Omega^1_{E/Y}$ For any nonnegative integer k we let

$$\mathscr{H}_k := Symm^k(\mathscr{H}) = \underline{\omega}^{-k} \oplus \underline{\omega}^{2-k} \oplus \cdots \oplus \underline{\omega}^{k}.$$

The Gauss-Manin connection $\nabla : \mathscr{H} \longrightarrow \mathscr{H} \otimes \Omega$ induces a connection

$$\nabla : \mathscr{H}_k \longrightarrow \mathscr{H}_k \otimes \Omega$$

for each integer $k \geq 0$, which we also call the Gauss-Manin connection.

The Deligne-Tate map ([**14**]) preserves Z_∞ and extends to a wide open neighborhood of Z_∞ *properly* contained in W_∞. Accordingly, the Gauss-Manin connection is endowed with a natural Frobenius structure over some sufficiently small wide open neighborhood of Z_∞. Katz spells out precisely how big this neighborhood can be, but this is a technical point that we will not need. It will be convenient to simplify the notation and write $Z_\infty^\dagger$ to denote a choice of such a wide open neighborhood of

Z_∞ with the additional property that the intersection of $Z_\infty^\dagger$ with any supersingular annulus is a concentric subannulus.

For k an integer, let $M_{k+2}^\dagger := \underline{\omega}^{k+2}(Z_\infty^\dagger)$ denote the space of overconvergent p-adic modular forms of weight $k+2$ and level $(Np,2)$ as before. For $k \geq 0$ we let

$$\kappa : M_{k+2}^\dagger \longrightarrow \mathscr{H}_k \otimes \Omega(Z_\infty^\dagger)$$

be the Kodaira Spencer map (see §4 of [**3**]). The canonical projection $\mathscr{H}_k \longrightarrow \underline{\omega}^{-k}$ induces a surjection $\mathscr{H}_k(Z_\infty^\dagger) \longrightarrow M_{-k}^\dagger$, and Coleman proves in [**3**] that there is a canonical $\mathbb{Q}_p$-linear section

$$\nu : M_{-k}^\dagger \longrightarrow \mathscr{H}_k(Z_\infty^\dagger)$$

satisfying the equation

$$\nabla(\nu(g)) = \kappa(\theta^{k+1}g)/k! \in \mathscr{H}_k \otimes \Omega(Z_\infty^\dagger)$$

for any $g \in M_{-k}$. Here $\theta^{k+1} : M_{-k}^\dagger \longrightarrow M_{k+2}^\dagger$ is the operator defined on q-expansions by

$$\theta^{k+1} : \sum_{n\geq 0} a_n q^n \longmapsto \sum_{n\geq 0} n^{k+1} a_n q^n.$$

For details, see Proposition 4.3 of [**3**].

Following Katz [**14**], Coleman [**2**] also defines a Frobenius structure on $Z_\infty^\dagger$ which gives rise to a "Frobenius operator" Φ acting on the cohomology of $\mathscr{H}_k$, $\underline{\omega}^k$, and Ω. Morevover, Φ commutes with $\nabla : \mathscr{H}_k \longrightarrow \mathscr{H}_k \otimes \Omega$ on $Z_\infty^\dagger$ (see §11 of [**2**]). On q-expansions of modular forms of weight k, Φ is given by $\Phi = p^k V$ where V is the operator on q-expansions given by $V(f)(q) = f(q^p)$, i.e.

$$V : \sum_{n\geq 0} a_n q^n \longmapsto \sum_{n\geq 0} a_n q^{np},$$

2. Coleman's $\mathscr{L}$-invariant.

In this section we recall Coleman's definition of the $\mathscr{L}$-invariant $\mathscr{L}(f)$ of a split multiplicative p-newform f of weight $k_0+2 \geq 2$. Let $\mathscr{H}_{k_0}^*$ denote the complex of sheaves associated to $\mathscr{H}_{k_0} \xrightarrow{\nabla} \mathscr{H}_{k_0} \otimes \Omega$ and consider the hypercohomology $\mathbb{H}^1(X, \mathscr{H}_{k_0}^*)$ with respect to the covering $\{W_\infty, W_0\}$ of X. The Hecke operators act on this space and the systems of eigenvalues that occur in it are the same as those that occur in the space of classical modular forms of weight k_0 and corresponding level. In particular, letting K be the field generated over $\mathbb{Q}_p$ by the eigenvalues of the Hecke operators acting on f, we obtain a $\mathbb{Q}_p$-subspace $H(f) \subseteq \mathbb{H}^1(X, \mathscr{H}_{k_0}^*)$ endowed with an action of the field K with the property that $H(f)$ is a 2-dimensional K-vector space on which the Hecke operators act as scalars according to the eigenvalues of f. Using his theory of p-adic integration, Coleman endows $H(f)$ with a natural monodromy module structure in which the monodromy is *non-trivial.* In [**15**], Mazur attaches an $\mathscr{L}$-invariant to any two dimensional monodromy module with non-trivial monodromy. Coleman's $\mathscr{L}$-invariant can be defined simply as the $\mathscr{L}$-invariant of Coleman's monodromy module.

We will use the more concrete definition that Coleman gives in [**2**]. For simplicity, we assume $k_0 > 0$ so that there are no nonzero horizontal sections of $\mathscr{H}_{k_0}$ defined on all of W_∞ nor on all of W_0, i.e. $H^0(W_\infty, \mathscr{H}_{k_0}^*) = H^0(W_0, \mathscr{H}_{k_0}^*) = 0$. On the other hand, one generally does find non-zero horizontal sections of $\mathscr{H}_{k_0}$ on the supersingular annuli $W = W_\infty \cap W_0$. Indeed, Coleman constructs two maps

$$\sigma, \rho : M_{k_0+2} \longrightarrow H^0(W, \mathscr{H}_{k_0}^*)$$

defined on the space M_{k_0+2} of classical modular forms of weight k_0+2 and appropriate level. The map σ is defined using Coleman integration (Definition 2.1 below) while the map ρ is defined in terms of residues (Definition 2.2).

Let $k \geq 0$ and $f \in M_{k+2}$ be a classical Hecke eigenform. Let α be the eigenvalue of the U-operator acting on f. We suppose $\alpha \neq 0$. The differential form $\omega_f := \kappa(f) \in \mathscr{H}_k \otimes \Omega(W_\infty)$ represents a cohomology class $[\omega_f] \in H^1(W_\infty, \mathscr{H}_k)$ and the Frobenius operator Φ acts on ω_f and also on $[\omega_f]$. Indeed, we have $\Phi([\omega_f]) = \frac{p^{k+1}}{\alpha} \cdot [\omega_f]$. Now Coleman's integration theory gives us a well-defined flabby antiderivative $I_\infty(f)$ defined on all of W_∞ which is rigid analytic on the ordinary residue disks, is log-analytic on the supersingular annuli and satisfies the following two properties

- $I_\infty(f)$ satisfies the differential equation

$$\nabla(I_\infty(f)) = \omega_f \quad \text{on } W_\infty.$$

- the flabby analytic section

$$I_\infty(f) - \frac{\alpha}{p^{k+1}} \Phi(I_\infty(f))$$

of $\mathscr{H}_k$ is rigid analytic on $Z_\infty^\dagger$ (i.e. not only on Z_∞, but also on some wide open neighborhood of Z_∞).

Similar considerations give rise to a well-defined flabby analytic section $I_0(f)$ of $\mathscr{H}_k$ over W_0 satisfying the differential equation

$$\nabla(I_0(f)) = \omega_f \quad \text{on } W_0.$$

Both $I_0(f)$ and $I_\infty(f)$ are defined on the overlap $W = W_\infty \cap W_0$. Coleman makes the following definition.

Definition 2.1. — *If $f \in M_{k+2}$ is a classical Hecke eigenform then we define $\sigma(f) \in H^0(W, \mathscr{H}_k^*)$ to be the horizontal section of $\mathscr{H}_k$ on W given by*

$$\sigma(f) := I_\infty(f)|_W - I_0(f)|_W.$$

The residue map $\rho : M_{k+2} \longrightarrow H^0(W, \mathscr{H}_k^*)$ is defined using the map

$$\text{Res} : \mathscr{H}_k \otimes \Omega(Z_\infty^\dagger) \longrightarrow H^0(W, \mathscr{H}_k^*)$$

which in turn is defined by $\text{Res}(\omega) :=$ the unique horizontal section of $\mathscr{H}_k$ on W whose restriction to $Z_\infty^\dagger \cap W$ is the residue of ω restricted to this disjoint union of oriented annuli. Note that here as elsewhere we use the standard orientation of the annuli, i.e. the orientation in which Z_∞ is *interior* to W.

Definition 2.2. — *Given $f \in M^{\dagger}_{k+2}$, we let $\omega_f := \kappa(f) \in \mathscr{H}_k \times \Omega(Z^{\dagger}_{\infty})$ and define* $\rho(f) := \mathrm{Res}(\omega_f)$.

Definition 2.3. — *Coleman's $\mathscr{L}$-invariant of a split multiplicative newform $f \in M_{k+2}$ is defined to be the unique element $\mathscr{L}(f) \in K$ for which $\sigma(f) = \mathscr{L}(f) \cdot \rho(f)$.*

The existence and uniqueness of such an $\mathscr{L}$-invariant was, of course, proved by Coleman (see [**2**]).

3. Families of modular forms

First of all we have the Eisenstein family. For each integer k there is an overconvergent p-adic modular form E_k of weight k whose q-expansion is given by

$$E_k := 1 + 2\zeta_p(1-k)^{-1} \sum_{k \geq 1} \sigma^*_{k+1}(n) q^n.$$

Here $\zeta_p(s)$ is the Kubota–Leopoldt p-adic zeta function and when $k = 0$ the above equality is understood to mean $E_0 = 1$. (Recall $\zeta_p(s)$ has a simple pole at $s = 1$). For integral $k \geq 0$ we set

$$t_k := \frac{1}{2}\zeta_p(1+k) \cdot E_{-k} \qquad \text{and} \qquad G_k := \frac{1}{2}\zeta_p(-1-k) \cdot E_{k+2}.$$

Then $t_k \in M^{\dagger}_{-k}$ is an overconvergent modular form of weight $-k$ and $G_k \in M_{k+2}$ is a classical modular form of weight $k+2$. The family t_k extends to a meromorphic family of Eisenstein series for $k \in \mathscr{X}$ with a simple pole at $k = 0$ and G_k defines a meromorphic family with a simple pole at $k = -2$. Moreover $G_k = t_{-2-k}$. The special point $k = 0$ will play a crucial role in the proof of Theorem B.

Proposition 3.1. — *The following assertions hold.*

1. *The family t_k, $k \in \mathscr{X}$, has a simple pole at $k = 0$ with residue given by*
$$\lim_{k \to 0} k t_k = \frac{1}{2} \cdot \left(1 - \frac{1}{p}\right).$$
2. *The residue of G_0 along any supersingular annulus is $1/2$:*
$$\rho(G_0) = -\frac{1}{2}.$$

Proof. — The first assertion is an immediate consequence of the well-known fact that the Kubota-Leopoldt p-adic zeta function $\zeta_p(s)$ has a simple pole at $s = 1$ and that the residue at $s = 1$ is given by

$$\lim_{s \to 1} (s-1)\zeta_p(s) = \left(1 - \frac{1}{p}\right).$$

To prove the second assertion, we first consider the special case $N = 1$. Then $\eta = \kappa(G_0)$ is a section of Ω over Y which extends to a meromorphic section over X with simple poles along the cusps. We want to compute $\mathrm{Res}(\eta) \in H^0(W)$. We remark first of all that since the eigenvalues of the Hecke operators acting on η are known, they are also known on $\text{Res}(\eta)$. Indeed, the eigenvalues are the same as those acting on constant functions on W. Hence $\text{Res}(\eta)$ is a constant. To determine what the constant is we use the fact that the sum of the residues along the oriented annuli at the "edges" of $W_\infty \setminus \{cusps\}$ is equal to zero. For the supersingular annuli this is the orientation we used above to define $\rho(f)$. However, the annuli around the cusps are orientated so that the cusps are *exterior* to the annuli, which is opposite the one used to compute the constant terms of Eisenstein series.

Under our assumption $N = 1$, there are a total of three cusps in W_∞ corresponding to the three cusps of $X(2)$. The constant terms of G_0 are the same at all of these cusps since G_0 is modular of level p. Since the natural map $X \longrightarrow X_0(p)$ is ramified of order 2 at each of these cusps and since the constant term of G_0 at the infinity cusp is $(1-p)/24$ we conclude that the sum of the residues along the oriented annuli around the cusps is $(p-1)/4$. Hence the sum of the residues along the supersingular annuli is $(1-p)/4$. But a simple calculation shows that the number of supersingular annuli in X is $(p-1)/2$. Hence the residue along any supersingular annulus is $-1/2$. This proves (2) when $N = 1$.

The general case follows at once since for arbitrary N, the map $X(Np, 2) \longrightarrow X(p, 2)$ is unramified over the supersingular annuli. This completes the proof of the proposition. □

We can remove Euler factors at p using the operator V on overconvergent modular forms defined on q-expansions by the formula $V(f)(q) = f(q^p)$. If F is an eigenform, then we let F^0 denote the eigenform obtained by removing the Euler factor at p. Thus, we have the families

$$\begin{aligned} t_k^0 &:= t_k - V(t_k) \\ G_k^0 &:= G_k - V(G_k) \\ f_k^0 &:= f_k - \alpha(k)V(f_k) \end{aligned}$$

For $k \geq 0$ we let $\eta_k := \kappa(G_k)$ and $\eta_k^0 := \kappa(G_k^0)$ where $\kappa : \mathscr{M}_{k+2} \longrightarrow \mathscr{H}_k \otimes \Omega$ is the Kodaira-Spencer map. We also set $g_k := \nu(t_k)$ and $g_k^0 := \nu(t_k^0)$. Then since $\theta^{k+1} t_k^0 = G_k^0$ it follows that

$$\nabla(g_k^0) = G_k^0.$$

Finally, for each integer $k \geq 0$ we may let $s_k := I_\infty(f_k)$ be the Coleman integral of f_k defined in section 1. Then s_k is a flabby section of $\mathscr{H}_k$ over W_∞. This section is characterized by the property that

$$s_k^0 := s_k - \frac{\alpha(k)}{p^{k+1}} \cdot \Phi(s_k)$$

is a rigid analytic section of $\mathscr{H}_k$ over $Z_\infty^\dagger$. Hence there is an overconvergent modular form $\phi_k^0 \in M_{-k}^\dagger$ such that

$$\nu(\phi_k^0) = s_k^0.$$

Hence $\theta^{k+1}(\phi_k^0) = f_k^0$. Finally, set

$$\begin{aligned} \omega_k &:= \kappa(f_k), \\ \omega_k^0 &:= \kappa(f_k^0). \end{aligned}$$

4. Some Pairings

As in the introduction, we fix an integer $k_0 \geq 0$. For each integer $k \geq 0$ cup product on the de Rham cohomology of the fibers of E/X induces a natural pairing $[\cdot,\cdot] : \mathscr{H}_k \times \mathscr{H}_{k+k_0} \longrightarrow \mathscr{H}_{k_0}$. This pairing induces natural pairings

$$\begin{aligned} [\cdot,\cdot] &: \mathscr{H}_k \times \mathscr{H}_{k+k_0} \otimes \Omega \longrightarrow \mathscr{H}_{k_0} \otimes \Omega; \\ [\cdot,\cdot] &: \mathscr{H}_k \otimes \Omega \times \mathscr{H}_{k+k_0} \longrightarrow \mathscr{H}_{k_0} \otimes \Omega. \end{aligned}$$

Proposition 4.1. — *These pairings satisfy the following identity for all $x \in \mathscr{H}_k$, and $y \in \mathscr{H}_{k+k_0}$*

$$\nabla[x, y] = [x, \nabla y] + [\nabla x, y].$$

Proof. — This follows from the product formula for differentiation. □

We will use a superscript † to denote overconvergent sections of a sheaf. For example, $\mathscr{H}_k^\dagger := \mathscr{H}_k(Z_\infty^\dagger)$. We may then define pairings

$$\begin{aligned} \langle\cdot,\cdot\rangle &: \mathscr{H}_k^\dagger \times \mathscr{H}_{k+k_0}^\dagger \otimes \Omega^\dagger \longrightarrow H^0(W, \mathscr{H}_{k_0}^*) \\ \langle\cdot,\cdot\rangle &: \mathscr{H}_k^\dagger \otimes \Omega^\dagger \times \mathscr{H}_{k+k_0}^\dagger \longrightarrow H^0(W, \mathscr{H}_{k_0}^*) \end{aligned}$$

by defining $\langle x, y\rangle := \mathrm{Res}([x, y])$ where $\mathrm{Res} : \mathscr{H}_{k_0}^\dagger \longrightarrow H^0(W, \mathscr{H}_{k_0}^*)$ is the residue map.

Recall the Hecke operators $U := U_p$ and w from §8 of [**2**]. On q-expansions of modular forms U is given by

$$U : \sum_{n\geq 0} a_n q^n \longmapsto \sum_{n\geq 0} a_{np} q^n.$$

The operator w acts on $\mathscr{H}_{k_0}(W)$ and satisfies $w^2 = p^{k_0}$ on this space. Hence

$$W_p := p^{-k_0/2} w$$

acts as an involution. Moreover, from §11 of [**2**] we have $\Phi = w$ on horizontal sections of $\mathscr{H}_{k_0}$ over W, hence $\Phi = p^{k_0/2} W_p$ on $H^0(W, \mathscr{H}_{k_0}^*)$.

Proposition 4.2. — *Let k, k_0 be non-negative integers and let $g \in M_{-k}^\dagger$, $f \in M_{k+k_0}^\dagger$, and $h \in M_{k+2}^\dagger$. Then the following assertions hold.*

1. *For $x = \nu(g) \in \mathscr{H}_k^\dagger$ and $\omega = \kappa(f) \in \mathscr{H}_{k+k_0}^\dagger \otimes \Omega^\dagger$ we have $\langle x, \Phi(\omega)\rangle = p^{k+\frac{k_0}{2}+1} \cdot W_p(\langle U(x), \omega\rangle)$;*
2. *For $\eta = \kappa(h) \in \mathscr{H}_k^\dagger \otimes \Omega^\dagger$ and $y \in \mathscr{H}_{k+k_0}^\dagger$ we have $\langle \eta, \Phi(y)\rangle = p^{k+\frac{k_0}{2}} \cdot W_p(\langle U(\eta), y\rangle)$.*

Proof. — Since $p^{k_0/2}W_p = \Phi$ and $p \cdot \Phi \circ Res = Res \circ \Phi$ we have

$$\begin{aligned} p^{k+\frac{k_0}{2}+1} \cdot W_p(\langle U(x), \omega\rangle) &= p^k \cdot \langle \Phi U(x), \Phi\omega\rangle \\ &= p^k \rho(\Phi U(g) \cdot \Phi(f)) \\ &= \rho(g \cdot \Phi(f)) \\ &= \langle x, \Phi(\omega)\rangle. \end{aligned}$$

This proves (1) and (2) is proved similarly. □

5. Proof of Theorem B

The operator W_p is an involution on $H^0(W, \mathscr{H}_{k_0})$. We let superscript + denote projection to the +-component under the action of W_p. Consider the function $\psi : \mathscr{X} \longrightarrow H^0(W, \mathscr{H}_{k_0})^+$ defined by

$$\psi(k) := \rho(t_k^0 f_{k+k_0}^0)^+ \in H^0(W, \mathscr{H}_{k_0})^+.$$

Since $t_k^0 f_{k+k_0}^0$ is an analytic family of overconvergent modular forms of weight k_0 we see at once that $\psi(k)$ is an analytic function of k defined on a neighborhood of 0 in $\mathscr{X}$. For the proof of Theorem B we will calculate $\psi(0)$ in two ways. First, by direct calculation we express $\psi(0)$ in terms of $\rho(f)$. Then we apply the product rule (Proposition 2) to express $\psi(0)$ in terms of $\sigma(f)$. Comparing these two expressions, Theorem B follows.

Define $u(k) := p^{-k_0/2} \cdot \alpha(k)$, the "unit part" of $\alpha(k)$.

Lemma 5.1. — *We have*

$$\psi(0) = -\frac{1}{2} \cdot \left(1 - \frac{1}{p}\right) \cdot u'(k_0) \cdot \rho(f).$$

Proof. — For an arbitrary integer $k \geq 0$ we have

$$\psi(k) = \rho\left(t_k^0 f_{k+k_0}^0\right)^+ = \langle g_k^0, \omega_{k+k_0}^0\rangle^+.$$

We also have

$$\begin{aligned} \langle g_k^0, \omega_{k+k_0}^0\rangle &= \langle g_k, \omega_{k+k_0}^0\rangle \\ &= \left\langle g_k,\ \omega_{k+k_0} - \tfrac{\alpha(k+k_0)}{p^{k+k_0+1}}\Phi(\omega_{k+k_0})\right\rangle \\ &= \langle g_k,\ \omega_{k+k_0}\rangle - \tfrac{\alpha(k+k_0)}{p^{k_0/2}} W_p(\langle U(g_k), \omega_{k+k_0}\rangle) \\ &= \langle g_k,\ \omega_{k+k_0}\rangle - u(k+k_0) \cdot W_p(\langle g_k, \omega_{k+k_0}\rangle). \end{aligned}$$

The first equality above follows from three facts: (1) $g_k^0 - g_k$ is in the image of Φ; (2) $\omega_{k+k_0}^0$ is in the kernel of U; and (3) the image of Φ is perpendicular to the kernel of U by Proposition 4.2. The third equality follows from Proposition 4.2(1). The last equality above follows from the fact that the Eisenstein series t_k is an eigenform for the U-operator with eigenvalue 1, hence $U(g_k) = g_k$.

Now project the above identity to the +-component for W_p to get

$$\begin{aligned}\psi(k) &= (1-u(k+k_0))\cdot\langle g_k,\omega_{k+k_0}\rangle^+\\ &= \frac{1-u(k+k_0)}{k}\cdot\rho(kt_kf_{k+k_0})^+.\end{aligned}$$

Letting $k\to 0$, using Propostion 3.1(1), and noting that $\rho(f)^+=\rho(f)$ we obtain

$$\psi(0) = -\frac{1}{2}\cdot\left(1-\frac{1}{p}\right)\cdot u'(k_0)\cdot\rho(f)$$

and the lemma is proved. □

Let $C_\infty := Z_\infty^\dagger\setminus Z_\infty$. Then C_∞ is a union of concentric annuli in the supersingular annuli. Note that the pairings $\langle x,y\rangle$ are well-defined so long as x,y are rigid on C_∞. In particular we have a well-defined pairing

$$\langle\cdot,\cdot\rangle : \Omega^1(C_\infty)\times\mathscr{H}_{k_0}(C_\infty)\longrightarrow\mathscr{H}_{k_0}(W)^\nabla.$$

defined by $\langle\omega,h\rangle=\mathrm{Res}_W(h\omega)$, where this latter is defined to be the unique horizontal section on W extending $\mathrm{Res}_{C_\infty}(h\omega)$.

We now turn to an application of Coleman's integration theory. In what follows, we will write a subscript $flog$ to denote flabby log-analytic sections of a rigid analytic sheaf. Such sections are, by definition, rigid analytic on the residue disks in the ordinary part of X and are log-analytic on the supersingular annuli. For details, see §10 of [**2**] and also [**1**].

Lemma 5.2. — *Let $e\in\mathscr{O}_{flog}(W_\infty)$ be any Coleman integral of η_0 (well-defined up to a constant). Restrict e to the supersingular annuli W and let $h=e-W_p(e)\in\mathscr{O}_{\log}(W)$. Let $z=h\cdot\rho(f)\in\mathscr{H}_{k_0,log}(W)$, and let $z^0:=z-p^{-1-k_0/2}\Phi(z)\in\mathscr{H}_{k_0,log}(C_\infty)$. Then z,z^0 have the following properties.*

1. *z^0 is rigid on C_∞.*
2. *$s_{k_0}+z$ is rigid on W.*
3. *$\langle\eta_0,z^0\rangle=0$.*
4. *$W_p(z)+z=0$ on the supersingular annuli W.*

Proof. — (1) Since e is a Coleman integral of η_0, we have $e^0:=e-p^{-1}\Phi(e)$ is rigid on $Z_\infty^\dagger$. Since $W_p(\eta_0)=-\eta_0$ on X, we have $W_p(e)+e$ is constant on W. It follows that $h^0:=h-p^{-1}\Phi(h)$ is also rigid on C_∞. On the other hand, $\Phi(\rho(f))=p^{k_0/2}\rho(f)$. Hence $z^0=h^0\cdot\rho(f)$, which is rigid on C_∞.

(2) By definition, $\nabla(s_{k_0})=\kappa(f)$. Hence, $\mathrm{Res}_W(\nabla(s_{k_0}))=\rho(f)$. On the other hand, $\mathrm{Res}_W(\nabla(z))=\mathrm{Res}_W(dh)\cdot\rho(f)$. But $dh=2\eta_0$ and we have shown in Proposition 3.1 that $\mathrm{Res}_W(\eta_0)=-1/2$, hence $\mathrm{Res}_W(\nabla(z))=\rho(f)$. We therefore have $\mathrm{Res}_W(\nabla(s_{k_0}+z)=0$ and it follows that $s_{k_0}+z$ is rigid on W, as claimed.

(3) We have $\langle\eta_0,z^0\rangle=\langle\eta_0,h^0\rangle\cdot\rho(f)$. Moreover, $\langle\eta_0,h^0\rangle=\langle\eta_0^0,h^0\rangle$ because the image of Φ is orthogonal to the kernel of U. But, $\langle\eta_0^0,h^0\rangle=\mathrm{Res}_W(h^0\eta_0^0)=\frac{1}{2}\mathrm{Res}_W(h^0dh^0)=0$, since h^0dh^0 is an exact differential on C_∞.

(4) Since $W_p(\rho(f))=\rho(f)$, this follows immediately from the definition of z.