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0 Introduction

Suppose X is a smooth and projective scheme over a perfect field k with Witt ring W. Mazur's fundamental theorem [23] establishes a striking link between the action of Frobenius and the Hodge filtration on the crystalline cohomology of X/W and suggests a close connection and analogy between F-crystals and Hodge structures. Applications of Mazur's theorem and its concomitant philosophy include Katz's conjecture on Newton polygons [*op. cit.*], a crystalline Torelli theorem for certain K3 surfaces [26], and a simple proof of the degeneration of the Hodge spectral sequence [7]. The theorem also underlies the deeper manifestations of the theory of p-adic periods, developed by Fontaine-Messing [12], Faltings [9], and Wintenberger [29].

Our main goal in this monograph is to formulate and prove a version of Mazur's theorem with coefficients in an F-crystal. In order to do this it is necessary to define and describe a "Hodge filtration" on an F-crystal and on its cohomology. This suggests our second goal, the development of a crystalline version of the notion of a complex variation of Hodge structure, which we call a "T-crystal" (the "T" is for transverse). These objects make sense on any formal scheme of finite type over W, especially for schemes smooth over W or over $W_\mu =: W/p^\mu W$ for $\mu \in \mathbf{Z}^+$. Putting together the "F" and the "T," we obtain the notion of "F-T-crystal," which we hope is a reasonable analog of a variation of Hodge structure, on any smooth complete scheme over W_μ. In particular, F-T-crystals of level one should correspond to p-divisible groups. [1] We show how to attach to a T-crystal (E', A) to

[1]This seems to follow easily from a recent result of Kato, which is essentially the case

a suitable F-crystal (E, Φ), and our formulation of Mazur's theorem relates the action of Frobenius on the cohomology of the F-crystal (E, Φ) to the corresponding Hodge filtration on the cohomology of (E', A).

This paper can be seen as a continuation of our previous study [24, 25] of Griffiths transversality and crystalline cohomology, here with emphasis on global, rather than local, results. An important feature of our approach is that we work systematically with the Hodge filtration on crystalline cohomology over W, not just its image in the De Rham cohomology over k.

Before describing the plan of our paper, it is helpful to begin by briefly reviewing the statement of Mazur's theorem. Let k be a perfect field of characteristic p and let W be its Witt ring, with F denoting the Frobenius automorphisms of k and of W. Then a nondegenerate F-crystal on k/W is a finitely generated free W-module E, together with an F-semi-linear injective endomorphism Φ. Even if k is algebraically closed, the classification of such objects is quite complicated, as is the case for Hodge structures. Now a Hodge structure can be greatly simplified by forgetting its integral lattice—one then just obtains a filtered vector space (H, Fil) determined up to isomorphism by the Hodge numbers $h^i =: \dim \mathrm{Gr}^i_{Fil} H$. Mazur's crystalline analog of this simplification is the passage from an F-crystal Φ to the associated F-span $\Phi \colon E' \to E$, in which one simply forgets that the source and target of Φ are one and the same W-module. It is easy to classify F-spans up to isomorphism. Namely, still following Mazur, we define a filtration on E by taking $M^i E' =: \Phi^{-1}(p^i E)$; it is then easy to to see that our span is determined up to isomorphism by the Hodge numbers of the filtered k-vector space $(E' \otimes k, M)$. Actually it turns out to be more convenient to work with a slightly different filtration A, given by $A^i E' =: \sum_j p^{[i-j]} M^j E'$, which in fact induces the same filtration as does M on $E' \otimes k$. This construction defines a functor $\alpha_{k/W}$ from the category of F-crystals on k/W to the category of filtered W-modules. We can now state Mazur's fundamental result [4, 8.41] in the following way: if we apply $\alpha_{k/W}$ to the canonical F-crystal structure on the crystalline cohomology of a suitable X/k, the resulting filtration A is just the Hodge filtration:

$$A^i E = H^n(X/W, J^{[i]}_{X/W}).$$

Now suppose that we have a *family* of F-crystals (E, Φ) on a smooth X/k, *i.e.*, an injective map of locally free crystals $\Phi \colon F^*_{X/W} E \to E$. Such an object is usually just called an "F-crystal on X/W," and we view it

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$\mu = 1$.

as an analog of a variation of Hodge structure. Similarly, one can view an F-span $\Phi: F^*_{X/W}E' \to E$ as an analog of a complex variation of Hodge structure. For each point x of X, one can perform Mazur's construction, and obtain a filtered $W(k(x))$-module $(E'(x), A(x))$. It turns out that these filtrations vary nicely in a family: they fit together to form a filtration A of the crystal E'. As in the complex case, A is not a filtration by subcrystals, but rather by sheaves in the crystalline topos, satisfying a crystalline version of Griffiths transversality. For example the filtration associated to the constant F-crystal is the filtration by the divided powers of the ideal $J_{X/W}$. We call the data (E', A) a "T-crystal," and thus we obtain a functor $\alpha_{X/W}$ from the category of F-spans on X/W to the category of T-crystals. It turns out that this functor is even an equivalence for crystals of level less than p. Now our generalization (*c.f.* (7.4.3) and (7.5.3)) of Mazur's theorem says that the functor α commutes with the formation of higher direct images, under suitable conditions.

The use of logarithmic structures in crystalline cohomology greatly increases its range of applicability, so we begin in Section 1 by reviewing and extending the theory of logarithmic crystals, due originally to Faltings [10], Fontaine and Illusie, Hyodo, and Kato [15], [20]. The main new features of our presentation are the systematic study of logarithmic differential operators and the theory of p-curvature and Cartier descent in a logarithmic context. This section may be of some foundational interest independent of the rest of the article. On the other hand, those readers who want to avoid the technical difficulties of logarithmic structures can omit it and just work with the trivial logarithmic structures throughout the rest of the paper.

The first real task in our program is the systematic study of Griffiths transversality in the crystalline setting. The advantage of this viewpoint, aside from its aesthetic appeal, is that it allows us to work in arithmetic and geometric directions simultaneously. The basic idea is the following: If (E, A) is a filtered $\mathcal{O}$-module over a ring $\mathcal{O}$ and J is an $\mathcal{O}$-ideal, we say that (E, A) is "G-transversal to J" if $JE \cap A^iE = JA^{i-1}E$ for every i. If (J, γ) is a divided power ideal ("PD-ideal"), this notion must be modified to read:

$$JE \cap A^iE = JA^{i-1}E + J^{[2]}A^{i-2}E + \cdots ;$$

we then say that (E, A) is "G-transversal to (J, γ)" or just "PD-transversal to J." Section 2 discusses this notion in detail, investigates its behavior under pullback, and establishes the technical and geometric underpinnings of our work.

We begin the study of crystals and Griffiths transversality *per se* in Section 3. Recall that if Y/W is smooth, there is an equivalence between the category of crystals of $\mathcal{O}_{Y/W}$-modules and the category $MIC(Y/W)$ of pairs (E_Y, ∇) consisting of a quasi-coherent sheaf E_Y of $\mathcal{O}_Y$-modules endowed with an integrable and p-adically nilpotent connection ∇. Consider the category of triples (E_Y, ∇, A_Y), where (E_Y, ∇) is an object of $MIC(Y/W)$ and A_Y is a filtration on E_Y which is Griffiths transversal to ∇. We shall see that this category is equivalent to the category of pairs (E, A), where E is as before a crystal of $\mathcal{O}_{Y/W}$-modules and A is a filtration of E by subsheaves in the crystalline site which is PD-transversal to the PD-ideal $J_{Y/W}$ of $\mathcal{O}_{Y/W}$. It is then easy to give a natural generalization of this condition for arbitrary schemes X/W (for example, for smooth schemes over k); we thus construct the category of T-crystals on X/W.

Section 4 develops the language and techniques that we shall use to interpolate the various filtrations that arise in our work on Mazur's theorem. For example, if (K, A, B) is a bifiltered object then for any subset σ of $\mathbf{Z} \times \mathbf{Z}$, we obtain a subobject $K_\sigma =: \sum\{A^i \cap B^j : (i, j) \in \sigma\}$. This defines a filtration of K indexed by the lattice of subsets of $\mathbf{Z} \times \mathbf{Z}$, and the correspondence $\sigma \mapsto K_\sigma$ is compatible with the lattice structure—a fact which plays a key technical role in our proofs. There is also a close connection between this lattice and the lattice of gauges ("1-gauges" in our terminology) considered by Mazur in [23]. After slightly modifying his notion of a "tame gauge structure," we discover a close connection between such structures and G-transversality. The section ends with a discussion of the cohomology of tame gauge structures, generalizing and simplifying the results of §2 and §3 of [23] and §8 of [4].

Section 5 prepares the way for our formulation of the generalization of Mazur's theorem. Suppose for simplicity that X is smooth over a perfect field k (and we are working with trivial log structures.) Instead of studying F-crystals, it is more natural and general to work with F-spans, *i.e.* p-isogenies $\Phi: F_X^* E' \to E$ in the category of crystals on X/W (*c.f.* (5.2.1)). We find a close connection between F-spans and T-crystals. Namely, we construct a functor $\alpha_{X/W}$ from the category of F-spans to the category of T-crystals, interpolating Mazur's construction of the filtration M on E' when X is a point. For spans of small level (or "width," *c.f.* (5.1.1)), this functor even turns out to be an equivalence of categories. We then introduce the notion of an "F-T-crystal" on a smooth lifting Y of X to W_μ; this is an F-crystal on X/W together with a lifting of its associated T-crystal to Y/W. The section ends with a discussion of the relationship between such F-T-crystals and the category MF^∇ of Fontaine-modules, including a simple proof of Faltings' structure theorem for Fontaine-modules.

Section 6 discusses the cohomology of T-crystals. It includes a filtered Poincaré lemma for T-crystals and some technical preparations that allow us to study bifiltered complexes and the associated hypercohomology spectral sequences. In particular, we show that T-crystals can often be "pushed forward." Thus if $f: X \to Y$ is a smooth proper morphism of smooth W_μ-schemes and if (E', A) is a T-crystal on X/W, the crystalline cohomology sheaves $R^q f_* E'$ inherit a T-crystal structure, under suitable hypotheses, *c.f.* (6.3.2). We take care to describe as carefully as possible the behavior of the Hodge filtration even when the dimension is large compared to p.

Section 7 is devoted to the formulation and proof of our analog of Mazur's theorem. The main formulation (7.3.1) of this theorem takes place on the level of complexes. We prove it by an unscrewing procedure based on the interpolation techniques of Section 4 until we are essentially reduced to a filtered version of the Cartier isomorphism. On the level of cohomology, our theorem asserts (7.4.3) that, with suitable hypotheses, the functor $\alpha_{X/W}$ commutes with higher direct images. This result allows us to show in (7.5.3) that (with suitable hypothesis), the higher direct images of F-T-crystals again form F-T-crystals. As this manuscript was nearing completion, I learned with great interest that Kazuya Kato [18] is working on a theory (cohomology of F-gauges), which is closely related to our treatment of Mazur's theorem, but uses a different point of view. (The original definition of F-gauges is due to Ekedahl, [8], and is inspired by work of Fontaine, Lafaille, Nygaard, and of course Mazur.)

Section 8 contains examples and applications of our theory. It begins with a very cursory discussion of liftings of T-crystals in mixed characteristic, leading to generalizations of the decomposition theorems of Deligne and Illusie [7] as well as vanishing theorems of Kodaira-type, all with coefficients in the Hodge complexes associated to an F-T-crystal. We also give a slight refinement (8.2.2) of a result of Faltings [9, IVb], which shows that the Hodge spectral sequence and torsion in crystalline cohomology are well-behaved, provided that the prime p is large compared to the dimension of the space and the width of the crystal. Next we discuss Hodge and Newton polygons associated to F-spans and F-crystals, and in particular establish a form of Katz's conjecture with coefficients in an F-crystal (fulfilling, at least partially, a hope expressed in [1]). One application of our use of logarithmic structures is the link we find between the mixed Hodge structure of a smooth variety in characteristic zero and the Newton polygon of its reduction modulo a suitable prime p (8.3.7). Finally we work out what our theory says about the cohomology of symmetric powers of F-T crystals on curves, with an eye