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# THE DISPLAY OF A FORMAL $p$-DIVISIBLE GROUP

*by*

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***Abstract.*** — We give a new Dieudonné theory which associates to a formal $p$-divisible group $X$ over an excellent $p$-adic ring $R$ an object of linear algebra called a display. On the display one can read off the structural equations for the Cartier module of $X$, and find the crystal of Grothendieck-Messing. We give applications to deformations of formal $p$-divisible groups.

## Introduction

We fix throughout a prime number $p$. Let $R$ be a commutative unitary ring. Let $W(R)$ be the ring of Witt vectors. The ring structure on $W(R)$ is functorial in $R$ and has the property that the Witt polynomials are ring homomorphisms:

$$\begin{aligned} \mathbf{w}_n : \qquad & W(R) \longrightarrow R \\ (x_0, \ldots x_i, \ldots) & \longmapsto x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^n x_n \end{aligned}$$

Let us denote the kernel of the homomorphism $\mathbf{w}_0$ by $I_R$. The Verschiebung is a homomorphism of additive groups:

$$\begin{aligned} {}^V : \qquad & W(R) \longrightarrow W(R) \\ (x_0, \ldots x_i, \ldots) & \longmapsto (0, x_0, \ldots x_i, \ldots) \end{aligned}$$

The Frobenius endomorphism ${}^F : W(R) \to W(R)$ is a ring homomorphism. The Verschiebung and the Frobenius are functorial and satisfy the defining relations:

$$\begin{aligned} &\mathbf{w}_n({}^F x) = \mathbf{w}_{n+1}(x), \quad \text{for } n \geq 0 \\ &\mathbf{w}_n({}^V x) = p\mathbf{w}_{n-1}(x), \text{ for } n > 0, \quad \mathbf{w}_0({}^V x) = 0. \end{aligned}$$

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Moreover the following relations are satisfied:

$$ {}^{FV} = p, \qquad {}^{V}({}^{F}xy) = x^{V}y, \quad x, y \in W(R) $$

We note that $I_R = {}^{V}W(R)$.

Let $P_1$ and $P_2$ be $W(R)$-modules. An ${}^{F}$-linear homomorphism $\phi : P_1 \to P_2$ is a homomorphism of abelian group which satisfies the relation $\phi(wm) = {}^{F}w\phi(m)$, where $m \in P$, $w \in W(R)$. Let

$$ \phi^{\sharp} : W(R) \otimes_{F,W(R)} P_1 \longrightarrow P_2 $$

be the linearization of $\phi$. We will call $\phi$ an ${}^{F}$-linear epimorphism respectively an ${}^{F}$-linear isomorphism if $\phi^{\sharp}$ is an epimorphism respectively an isomorphism.

The central notion of these notes is that of a display. The name was suggested by the displayed structural equations for a reduced Cartier module introduced by Norman [N]. In this introduction we will assume that $p$ is nilpotent in $R$.

***Definition 1.*** — A $3n$-display over $R$ is a quadruple $(P, Q, F, V^{-1})$, where $P$ is a finitely generated projective $W(R)$-module, $Q \subset P$ is a submodule and $F$ and $V^{-1}$ are ${}^{F}$-linear maps $F : P \to P$, $\quad V^{-1} : Q \to P$.

The following properties are satisfied:

(i) $I_R P \subset Q \subset P$ and $P/Q$ is a direct summand of the $W(R)$-module $P/I_R P$.

(ii) $V^{-1} : Q \longrightarrow P$ is a ${}^{F}$-linear epimorphism.

(iii) For $x \in P$ and $w \in W(R)$, we have

$$ V^{-1}({}^{V}wx) = wFx. $$

If we set $w = 1$ in the relation (iii) we obtain:

$$ Fx = V^{-1}({}^{V}1x) $$

One could remove $F$ from the definition of a 3n-display. But one has to require that the ${}^{F}$-linear map defined by the last equation satisfies (iii).

For $y \in Q$ one obtains:

$$ Fy = p \cdot V^{-1}y $$

We note that there is no operator $V$. The reason why we started with $V^{-1}$ is the following example of a 3n-display. Let $R = k$ be a perfect field and let $M$ be a Dieudonné module. It is a finitely generated free $W(k)$-module which is equipped with operators $F$ and $V$. Since $V$ is injective, there is an inverse operator $V^{-1} : VM \to M$. Hence one obtains a display $(M, VM, F, V^{-1})$. In fact this defines an equivalence of the category of Dieudonné modules with the category of 3n-displays over $k$.

Let us return to the general situation. The $W(R)$-module $P$ always admits a direct decomposition

$$ P = L \oplus T, $$

such that $Q = L \oplus I_R T$. We call it a normal decomposition. For a normal decomposition the following map is a ${}^F$-linear isomorphism:

$$V^{-1} \oplus F : L \oplus T \longrightarrow P$$

Locally on $\operatorname{Spec} R$ the $W(R)$-modules $L$ and $T$ are free. Let us assume that $T$ has a basis $e_1, \ldots, e_d$ and $L$ has a basis $e_{d+1}, \ldots, e_h$. Then there is an invertible matrix $(\alpha_{ij})$ with coefficients in $W(R)$, such that the following relations hold:

$$Fe_j = \sum_{i=1}^{h} \alpha_{ij} e_i, \quad \text{for } j = 1, \ldots, d$$

$$V^{-1} e_j = \sum_{i=1}^{h} \alpha_{ij} e_i \quad \text{for } j = d+1, \ldots, h$$

Conversely for any invertible matrix $(\alpha_{ij})$ these relations define a 3n-display.

Let $(\beta_{kl})$ the inverse matrix of $(\alpha_{ij})$. We consider the following matrix of type $(h-d) \times (h-d)$ with coefficients in $R/pR$:

$$B = (\mathbf{w}_0(\beta_{kl}) \text{ modulo } p)_{k,l=d+1,\ldots,h}$$

Let us denote by $B^{(p)}$ be the matrix obtained from $B$ by raising all coefficients of $B$ to the power $p$. We say that the 3n-display defined by $(\alpha_{ij})$ satisfies the $V$-nilpotence condition if there is a number $N$ such that

$$B^{(p^{N-1})} \cdots B^{(p)} \cdot B = 0.$$

The condition depends only on the display but not on the choice of the matrix.

***Definition 2.*** — A 3n-display which locally on $\operatorname{Spec} R$ satisfies the $V$-nilpotence condition is called a display.

The 3n-display which corresponds to a Dieudonné module $M$ over a perfect field $k$ is a display, iff $V$ is topologically nilpotent on $M$ for the $p$-adic topology. In the covariant Dieudonné theory this is also equivalent to the fact that the $p$-divisible group associated to $M$ has no étale part.

Let $S$ be a ring such that $p$ is nilpotent in $S$. Let $\mathfrak{a} \subset S$ be an ideal which is equipped with divided powers. Then it makes sense to divide the Witt polynomial $\mathbf{w}_m$ by $p^m$. These divided Witt polynomials define an isomorphism of additive groups:

$$W(\mathfrak{a}) \longrightarrow \mathfrak{a}^{\mathbb{N}}$$

Let $\mathfrak{a} \subset \mathfrak{a}^{\mathbb{N}}$ be the embedding via the first component. Composing this with the isomorphism above we obtain an embedding $\mathfrak{a} \subset W(\mathfrak{a})$. In fact $\mathfrak{a}$ is a $W(S)$-submodule of $W(\mathfrak{a})$, if $\mathfrak{a}$ is considered as a $W(S)$-module via $\mathbf{w}_0$. Let $R = S/\mathfrak{a}$ be the factor ring. We consider a display $\widetilde{\mathcal{P}} = (\widetilde{P}, \widetilde{Q}, \widetilde{F}, \widetilde{V}^{-1})$ over $S$. By base change we obtain a display over $R$:

$$\widetilde{\mathcal{P}}_R = \mathcal{P} = (P, Q, F, V^{-1})$$

By definition one has $P = W(R) \otimes_{W(S)} \widetilde{P}$. Let us denote by $\widehat{Q} = W(\mathfrak{a})\widetilde{P} + \widetilde{Q} \subset \widetilde{P}$ the inverse image of $Q$. Then we may extend the operator $\widetilde{V}^{-1}$ uniquely to the domain of definition $\widehat{Q}$, such that the condition $\widetilde{V}^{-1}\mathfrak{a}\widetilde{P} = 0$ is fulfilled.

***Theorem 3.*** — *With the notations above let $\widetilde{\mathcal{P}}' = (\widetilde{P}', \widetilde{Q}', \widetilde{F}, \widetilde{V}^{-1})$ be a second display over $S$, and $\mathcal{P}' = (P', Q', F, V^{-1})$ the display over $R$ obtained by base change. Assume we are given a morphism of displays $u : \mathcal{P} \to \mathcal{P}'$ over $R$. Then $u$ has a unique lifting $\widetilde{u}$ to a morphism of quadruples:*

$$\widetilde{u} : (\widetilde{P}, \widehat{Q}, \widetilde{F}, \widetilde{V}^{-1}) \longrightarrow (\widetilde{P}', \widehat{Q}', \widetilde{F}, \widetilde{V}^{-1}).$$

This allows us to associate a crystal to a display: Let $R$ be a ring, such that $p$ is nilpotent in $R$. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over $R$. Consider a surjection $S \to R$ whose kernel $\mathfrak{a}$ is equipped with a divided power structure. If $p$ is nilpotent in $S$ we call such a surjection a pd-thickening of $R$. Let $\widetilde{\mathcal{P}} = (\widetilde{P}, \widetilde{Q}, \widetilde{F}, \widetilde{V}^{-1})$ be any lifting of the display $\mathcal{P}$ to $S$. By the theorem the module $\widetilde{P}$ is determined up to canonical isomorphism by $\mathcal{P}$. Hence we may define:

$$\mathcal{D}_{\mathcal{P}}(S) = S \otimes_{W(S)} \widetilde{P}$$

This gives a crystal on $\operatorname{Spec} R$ if we sheafify the construction.

Next we construct a functor $BT$ from the category of 3n-displays over $R$ to the category of formal groups over $R$. A nilpotent $R$-algebra $\mathcal{N}$ is an $R$-algebra (without unit), such that $\mathcal{N}^N = 0$ for a sufficiently big number $N$. Let $\mathrm{Nil}_R$ denote the category of nilpotent $R$-algebras. We will consider formal groups as functors from the category $\mathrm{Nil}_R$ to the category of abelian groups. Let us denote by $\widehat{W}(\mathcal{N}) \subset W(\mathcal{N})$ the subgroup of all Witt vectors with finitely many nonzero components. This is a $W(R)$-submodule. We consider the functor $\mathbf{G}^0_{\mathcal{P}}(\mathcal{N}) = \widehat{W}(\mathcal{N}) \otimes_{W(R)} P$ on $\mathrm{Nil}_R$ with values in the category of abelian groups. Let $\mathbf{G}^{-1}_{\mathcal{P}}$ be the subgroup functor which is generated by all elements in $\widehat{W}(\mathcal{N}) \otimes_{W(R)} P$ of the following form:

$${}^V\xi \otimes x, \quad \xi \otimes y, \qquad \xi \in \widehat{W}(\mathcal{N}),\ y \in Q,\ x \in P.$$

Then we define a map:

$$V^{-1} - \mathrm{id} : \mathbf{G}^{-1}_{\mathcal{P}} \longrightarrow \mathbf{G}^0_{\mathcal{P}} \tag{1}$$

On the generators above the map $V^{-1} - \mathrm{id}$ acts as follows:

$$(V^{-1} - \mathrm{id})({}^V\xi \otimes x) = \xi \otimes Fx - {}^V\xi \otimes x$$
$$(V^{-1} - \mathrm{id})(\xi \otimes y) = {}^F\xi \otimes V^{-1}y - \xi \otimes y$$

***Theorem 4.*** — *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over $R$. The cokernel of the map (1) is a formal group $BT_{\mathcal{P}}$. Moreover one has an exact sequence of functors on $\mathrm{Nil}_R$:*

$$0 \longrightarrow \mathbf{G}^{-1}_{\mathcal{P}} \xrightarrow{V^{-1} - \mathrm{id}} \mathbf{G}^0_{\mathcal{P}} \longrightarrow BT_{\mathcal{P}} \longrightarrow 0$$