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Matthew Emerton, Mark Kisin

Abstract. — Let $\mathbb{F}_q$ denote the finite field of order q (a power of a prime p), let X be a smooth scheme over a field k containing $\mathbb{F}_q$, and let Λ be a finite $\mathbb{F}_q$-algebra. We study the relationship between constructible Λ-sheaves on the étale site of X, and a certain class of quasi-coherent $\mathscr{O}_X \otimes_{\mathbb{F}_q} \Lambda$-modules equipped with a "unit" Frobenius structure. We show that the two corresponding derived categories are anti-equivalent as triangulated categories, and that this anti-equivalence is compatible with direct and inverse images, tensor products, and certain other operations.

We also obtain analogous results relating complexes of constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on smooth $W_n(k)$-schemes, and complexes of Berthelot's arithmetic $\mathscr{D}$-modules, equipped with a unit Frobenius.

***Résumé* (La correspondance de Riemann-Hilbert pour les F-cristaux unités)**

Soit $\mathbb{F}_q$ le corps à q éléments (où q est une puissance d'un nombre premier p), soit X un schéma lisse sur un corps k contenant $\mathbb{F}_q$ et soit Λ une $\mathbb{F}_q$-algèbre finie. Nous étudions la relation entre les Λ-faisceaux constructibles sur le site étale de X et une certaine classe de $\mathscr{O}_X \otimes_{\mathbb{F}_q} \Lambda$-modules quasi-cohérents munis d'une structure de Frobenius « unité ». Nous montrons que les deux catégories dérivées correspondantes sont anti-équivalentes comme catégories triangulées et que cette anti-équivalence est compatible aux images directes et inverses, aux produits tensoriels et à certaines autres opérations.

Nous obtenons également des résultats du même type reliant complexes de $\mathbb{Z}/p^n\mathbb{Z}$-faisceaux constructibles sur les $W_n(k)$-schémas lisses et complexes de $\mathscr{D}$-modules arithmétiques de Berthelot munis de Frobenius unité.

CONTENTS

GENERAL INTRODUCTION

Let X be a smooth complex analytic space. One knows that there is an equivalence of categories between the category of local systems of $\mathbb{C}$-vector spaces on X and the category of coherent $\mathcal{O}_X$-modules equipped with an integrable connection ∇. A serious defect of this theory is that the category of local systems, and similarly that of modules with connection, is not stable under taking direct images. For example, if $f : Y \to X$ is a proper map of smooth complex analytic spaces, then the higher direct images $R^i f_* \mathbb{C}$ are guaranteed to be a local system only over the points where f is smooth. Similarly, the connection on the relative De Rham cohomology may be singular at the points where f is not smooth. This defect is remedied by the theory of $\mathscr{D}$-modules, which shows that there is a relationship between the category of constructible sheaves, and the category of regular holonomic $\mathscr{D}$-modules. More precisely, the two corresponding derived categories are equivalent as triangulated categories, and this equivalence respects the "six operations" $f^!$, $f_!$, f^*, f_*, $\underline{RHom}^\bullet$, and $\overset{\mathbb{L}}{\otimes}$. This result, originally proved by Kashiwara and by Mebkhout with a later algebraic approach due to Beilinson and Bernstein, is known as the "Riemann-Hilbert correspondence".

The purpose of this paper is to study a certain characteristic p analogue of the above situation. Our starting point is a theorem of Nick Katz [**Ka2**, Prop. 4.1.1]

***Theorem* (Katz).** — *Let k be a perfect field of characteristic p containing $\mathbb{F}_q$, where $q = p^r$. If X is a smooth scheme over $W_n(k)$, the ring of Witt vectors of k of length n, and if F_X is a lift to X of the Frobenius on $W_n(k)$, then there is an equivalence of categories between the category of locally free étale sheaves of $W_n(\mathbb{F}_q)$-modules $\mathcal{L}$, and the category of coherent, locally free $\mathcal{O}_X$-modules $\mathcal{E}$ equipped with an $\mathcal{O}_X$-linear isomorphism $(F_X^r)^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. This equivalence is realised by associating $\mathcal{E} = \mathcal{L} \otimes_{W_n(k)} \mathcal{O}_X$ to $\mathcal{L}$.*

In the context of this paper, Katz's theorem should be regarded as the analogue of the relation between local systems and vector bundles with connection. The main purpose of this volume is to extend Katz's result to a Riemann-Hilbert type correspondence, first when X is actually a smooth k-scheme (§§1–12), and then more generally, for smooth $W_n(k)$ schemes (§§13–17). Each of these two parts of the paper has its own introduction, which provides a detailed outline of its contents. The remainder of this general introduction is devoted to explaining our Riemman-Hilbert correspondence, and some of its applications, in more detail. Let us also point out that the paper [**EK1**] provides a summary of the main results and key techniques of this volume.

To explain our results, suppose that X is a smooth $W_n(k)$-scheme with a lift of Frobenius F_X, as in Katz's theorem. We begin by introducing the notion of a *locally finitely generated unit* $\mathcal{O}_{F^r,X}$-module. If r is a fixed positive integer, then a *unit* $\mathcal{O}_{F^r,X}$-module on X is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ equipped with an isomorphism $(F_X^r)^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$. This isomorphism endows $\mathcal{M}$ with the structure of a sheaf of left modules over the sheaf of rings $\mathcal{O}_{F^r,X} = \mathcal{O}_X[F^r]$, where $\mathcal{O}_X[F^r]$ denotes the twisted polynomial ring given by the relation $F^r a = a^q F^r$, for any section a of $\mathcal{O}_X$. The unit $\mathcal{O}_{F^r,X}$-module $\mathcal{M}$ is said to be *locally finitely generated* if, locally on X, it is finitely generated as a left $\mathcal{O}_{F^r,X}$-module.

When X is a smooth k-scheme, the main result of this paper generalises Katz's theorem to an (anti-) equivalence of two triangulated categories: the derived category $D_c^b(X_{\acute{e}t})$ of bounded complexes of étale sheaves of $\mathbb{F}_q$-modules whose cohomology sheaves are constructible, and the derived category $D_{lfgu}^b(\mathcal{O}_{F^r,X})$ of bounded complexes of left $\mathcal{O}_{F^r,X}$-modules (on the Zariski site of X) whose cohomology sheaves are locally finitely generated unit $\mathcal{O}_{F^r,X}$-modules.

If we let $\pi_X : X_{\acute{e}t} \to X$ denote the natural morphism from the étale site of X to the Zariski site of X, then this anti-equivalence is given by associating

$$\mathcal{M}^\bullet = \mathrm{M}(\mathcal{F}^\bullet) = R\pi_{X*}\underline{RHom}^\bullet_{\mathbb{F}_q}(\mathcal{F}^\bullet, \mathcal{O}_X)[d_X]$$

to a complex $\mathcal{F}^\bullet$ in $D_c^b(X_{\acute{e}t})$, and associating

$$\mathcal{F}^\bullet = \mathrm{Sol}(\mathcal{M}^\bullet) = \underline{RHom}^\bullet_{\mathcal{O}_{F^r,X}}(\pi_X^*\mathcal{M}^\bullet, \mathcal{O}_{X_{\acute{e}t}})[d_X]$$

to a complex $\mathcal{M}^\bullet$ in $D_{lfgu}^b(\mathcal{O}_{F^r,X})$. (Here d_X denotes the (locally constant) dimension of X.) A comment on the choice of notation: M is intended to suggest a Dieudonné module functor, while Sol is borrowed from the theory of $\mathscr{D}$-modules.

When X is a smooth $W_n(k)$-scheme, the sheaf of rings $\mathcal{O}_{F^r,X}$ is replaced by a more intricate sheaf of rings $\mathscr{D}_{F,X}$, obtained by adjoining to $\mathcal{O}_X$ the differential operators of all orders on X, together with a "local lift of Frobenius." (When $n > 1$ we consider only the case $r = 1$.) A pleasing point is that the sheaf $\mathscr{D}_{F,X}$ that one obtains is independent of the choice of local lift, as any two lifts are congruent modulo p, and so either lift may be expressed as a polynomial combination of the other lift and appropriate differential operators. Thus (unlike in Katz's result stated above) we are not restricted to the consideration of smooth $W_n(k)$-schemes which admit a global lift of Frobenius. Another important technical result is that locally the sheaf $\mathcal{O}_X$ has a finite length resolution by free left $\mathscr{D}_{F,X}$-modules, whereas this is false if one works just with differential operators, and does not adjoin the Frobenius lifts. Thus the differential operators and lifts of Frobenius complement one another.

A sheaf of $\mathscr{D}_{F,X}$-modules $\mathcal{M}$ is called *locally finitely generated unit*, if it is quasi-coherent over $\mathcal{O}_X$, is finitely generated over $\mathscr{D}_{F,X}$ locally on X, and if the maps $F^*\mathcal{M} \to \mathcal{M}$ defined by local lifts of Frobenius are isomorphisms. The Riemann-Hilbert correspondence then asserts an anti-equivalence between two triangulated categories. One is the category $D^b_{ctf}(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$, which is the full subcategory of the bounded derived category of étale sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules, consisting of complexes with constructible cohomology, and finite Tor-dimension. The other is the category $D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$, which is the full subcategory of the bounded derived category of $\mathscr{D}_{F,X}$-modules, consisting of complexes with locally finitely generated unit cohomology sheaves, and finite $\mathcal{O}_X$-Tor dimension. The definition of the anti-equivalence is given by the same formulas as for $\mathcal{O}_{F,X}$-modules, except that $\mathcal{O}_{F,X}$ is replaced by $\mathscr{D}_{F,X}$ in the definition of Sol.

When $n = 1$, there is an equivalence between $D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$ and $D^b_{lfgu}(\mathcal{O}_{F,X})$, and one recovers the previous correspondence.

The correspondence we construct is compatible with three of Grothendieck's six operations, namely the operations $f_!$, f^{-1}, and $\overset{\mathbb{L}}{\otimes}$ on $D^b_c(X_{\acute{e}t})$. (Of course, for this claim of compatibility to have any sense, one must have an *a priori* description of the corresponding triple of operators on $D^b_{lfgu}(\mathcal{O}_{F^r,X})$ and $D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$. A significant part of our work is devoted to constructing such operations for left $\mathcal{O}_{F^r,X}$-modules and $\mathscr{D}_{F,X}$-modules and establishing their main properties.) This may at first seem disappointing, but in fact these are the only three which are defined in our situation. Indeed for complexes of p-torsion étale sheaves the push-forward f_* does not preserve the property of having constructible cohomology sheaves. Also one cannot define $f^!$, since there is no duality (for example because the purity theorem fails horribly). This lack of a duality is also the reason that we construct an anti-equivalence rather than an equivalence of categories.

As already remarked, when $n = 1$ we develop a somewhat more elaborate theory than in the general case. For example we show that, as a (perhaps weak) compensation for the absense of three of the six opearations, our correspondence relates two other operations which have no analogue over $\mathbb{C}$. Namely, if $q' = p^{r'}$, and $\mathbb{F}_q \subset \mathbb{F}_{q'} \subset k$, then the functor "$\mathbb{F}_{q'} \otimes_{\mathbb{F}_q} -$" converts sheaves of $\mathbb{F}_q$-vector spaces into sheaves of $\mathbb{F}_{q'}$-vector spaces. Conversely the functor "pass to the underlying $\mathbb{F}_q$-structure" converts sheaves of $\mathbb{F}_{q'}$-vector spaces into sheaves of $\mathbb{F}_q$-vector spaces. We call these operations respectively *induction* and *restriction*. There are corresponding operations for left $\mathcal{O}_{F^r,X}$-modules, namely "$\mathcal{O}_{F^r,X} \otimes_{\mathcal{O}_{F^{r'},X}} -$" and "pass to the underlying $\mathcal{O}_{F^{r'},X}$-module structure", which we also call induction and restriction. We show that our anti-equivalence interchanges the operations of induction and restriction.