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p-ADIC HODGE THEORY AND VALUES OF ZETA FUNCTIONS OF MODULAR FORMS

by

Kazuya Kato

Abstract. — If f is a modular form, we construct an Euler system attached to f from which we deduce bounds for the Selmer groups of f. An explicit reciprocity law links this Euler system to the p-adic zeta function of f which allows us to prove a divisibility statement towards Iwasawa's main conjecture for f and to obtain lower bounds for the order of vanishing of this p-adic zeta function. In particular, if f is associated to an elliptic curve E defined over $\mathbb{Q}$, we prove that the p-adic zeta function of f has a zero at $s = 1$ of order at least the rank of the group of rational points on E.

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Introduction

One of the most fascinating subjects in number theory is the study of mysterious relations between zeta functions and "arithmetic groups". Here "arithmetic groups" include ideal class groups of number fields, Mordell-Weil groups of abelian varieties over number fields, Selmer groups associated to Galois representations of number fields, etc., which play important roles in number theory. Among such relations, we have Iwasawa theory (relation between zeta functions and ideal class groups) which is a refinement in 20th century of the class number formula in 19th century, Birch Swinnerton-Dyer conjectures (relation between zeta functions and Mordell-Weil groups), etc., and much of such relations are still conjectural. When we study such relations, a big difficulty is that zeta functions and arithmetic groups are too much different in nature; zeta functions are analytic and arithmetic groups are algebraic and it is very difficult to understand why they are closely related.

After Kolygavin, it was recognized that zeta functions have not only the usual analytic shapes (Euler products), but also arithmetic shapes (Euler systems), and that it is useful to consider these arithmetic shapes for the study of relations between zeta functions and arithmetic groups; it is more easy to understand the relation between the arithmetic shapes of zeta functions and arithmetic groups which are not far in nature, than the relation of analytic shapes and arithmetic groups.

zeta function (analytic)	$\xleftrightarrow{\text{too far}}$	arithmetic groups (algebraic)
Euler systems (= arithmetic shapes of zeta functions)	$\xleftrightarrow{\text{near}}$	arithmetic groups (algebraic)

In this paper, by considering the Euler systems of Beilinson elements in K_2 of modular curves, which are regarded as "arithmetic shapes" of zeta functions of elliptic modular forms, and by using p-adic Hodge theory, we obtain results on the relations between zeta functions of elliptic modular forms and Selmer groups associated to modular forms, and results in Iwasawa theory of modular forms.

Since it is now known that all elliptic curves over $\mathbb{Q}$ are modular ([**Wi**] [**BCDT**]), this gives also results on Birch Swinnerton-Dyer conjectures for elliptic curves over $\mathbb{Q}$.

The main results of this paper are the following. (Please see the text for the precise statements.)

Theorem. — *Let f be an eigen cusp form for $\Gamma_1(N)$ of weight $k \geqslant 2$.*

(1) *(Thm. 14.2) Let $r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$, and assume $r \neq k/2$. Then for any finite abelian extension* K *of $\mathbb{Q}$, the Selmer group* $\mathrm{Sel}(\mathrm{K}, f, r)$ *of f over* K *with r twist is a finite group.*

(2) *(Thm. 14.2) Assume k is even. Let K be a finite abelian extension of $\mathbb{Q}$. Let $\chi : \mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{C}^\times$ be a character, and assume $\mathrm{L}(f, \chi, k/2) \neq 0$. Then the χ-part* $\mathrm{Sel}(\mathrm{K}, f, k/2)^{(\chi)}$ *of* $\mathrm{Sel}(\mathrm{K}, f, k/2)$ *is a finite group.*

(3) *(Thm. 18.4) Assume k is even. Then*

$$p\text{-adic corank of } \mathrm{Sel}\left(\mathrm{K}, f, k/2\right) \leqslant \mathrm{ord}_{s=k/2} \quad (p\text{-adic zeta function of } f).$$

(4) *(Thm. 17.4) Assume f is good ordinary at p. Then*

$$\mathfrak{X} \underset{\mathrm{def}}{=} \mathrm{Hom}(\varinjlim_n \mathrm{Sel}(\mathbb{Q}(\zeta_{p^n}), f, r), (\mathbb{Q}_p/\mathbb{Z}_p)(r))$$

for $1 \leqslant r \leqslant k-1$ is independent of r and the characteristic ideal of $\mathfrak{X}$ divides p^n times the p-adic zeta function of f for some $n \geqslant 0$.

In some cases, we can drop p^n in (4) (Thm 17.4 (3)). This (4) is a partial answer to a conjecture of Greenberg ([**Gr1**], the case of elliptic curves was conjectured by Mazur [**Ma1**]) who predicts the equality in place of divisibility in (4). We also obtain results on "Iwasawa main conjecture for modular forms without p-adic zeta functions" (Thm. 12.5) and results on Tamagawa number conjectures ([**BK2**]) for modular forms (Thm. 14.5).

There are already many results on these subjects (for example, [**BD**], [**CW**], [**Ru2**], [**Ko**], [**Ne**], ...). Most of former works use elliptic units and Heegner points as "arithmetic shapes of zeta functions", whereas we use Beilinson elements instead. The part of the above Theorem concerning eigen cusp forms f with complex multiplication depends on results of [**Ru2**] on main conjectures of imaginary quadratic fields.

The plan of this paper is as follows. In Chapter I, we define Euler systems of Beilinson elements in K_2 of modular curves (§ 2) and also Euler systems in the spaces of modular forms (§ 4). The former Euler systems are related to $\lim_{s\to 0} s^{-1}\,\mathrm{L}(f, s)$ for cusp forms f of weight 2 by the theory of Beilinson, and the latter Euler systems are related to the zeta values $\mathrm{L}(f, r)$ ($r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$) of cusp forms f of weight

$k \geqslant 2$ by the theory of Shimura. In Chapter 2, by using the above Euler systems in K_2 of modular forms, we define p-adic Euler systems in the Galois cohomology of p-adic Galois representations associated to eigen cusp forms of weight $\geqslant 2$ (§ 8). We prove that via p-adic Hodge theory, these p-adic Euler systems are closely related to the Euler systems in the space of modular forms (§ 9), and hence closely related to the zeta values $L(f,r)$ ($r \in \mathbb{Z}, 1 \leqslant r \leqslant k-1$) for cusp forms of weight $k \geqslant 2$. In chapter III and Chapter IV, by using this relation of our p-adic Euler systems with zeta values, and by using the general theory of Euler systems in Galois cohomology, we obtain our main results.

A large part of results of this paper in the case of modular forms of weight 2 were introduced in Scholl [**Sc2**] and Rubin [**Ru3**].

This work is a continuation of my joint work with S. Bloch on Tamagawa numbers of motives ([**BK2**]), and I am very thankful to him for his great influences. I express my sincere gratitude to J. Coates, M. Kurihara, and T. Saito for their constant encouragements in my writing this paper. I am thankful to N. Kurokawa for teaching me modular forms and Rankin convolutions. I am also thankful to J. Coates, G. Faltings, M. Flach, H. Hida, N. Katz, M. Kurihara, B. Mazur, T. Shioda, T. Tsuji, A. Wiles, for advice, and to P. Colmez for corrections on the manuscript.

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CHAPTER I

EULER SYSTEMS IN K_2 OF MODULAR CURVES AND EULER SYSTEMS IN THE SPACES OF MODULAR FORMS

In this Chapter I, we consider Euler systems in K_2 of modular curves (§ 2) and Euler systems in the spaces of modular forms (§ 4). The former (resp. latter) come from the work of Beilinson [**Be**] (resp. Shimura [**Sh**]) and are related to the zeta values $\lim_{s\to 0} s^{-1}L(f,s)$ (resp. $L(f,r)$ $(1 \leqslant r \leqslant k-1)$) for cusp forms f of weight 2 (resp. k), by the theory in [**Be**] (resp. [**Sh**]).

§ 1 is a review on Siegel units (resp. Eisenstein series) and is a preparation for § 2 (resp. § 4).