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GENERALIZED BIALGEBRAS  
AND TRIPLES OF OPERADS

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# GENERALIZED BIALGEBRAS AND TRIPLES OF OPERADS

Jean-Louis Loday

**Abstract.** — We introduce the notion of generalized bialgebra, which includes the classical notion of bialgebra (Hopf algebra) and many others, like, for instance, the tensor algebra equipped with the deconcatenation as coproduct. We prove that, under some mild conditions, a connected generalized bialgebra is completely determined by its primitive part. This structure theorem extends the classical Poincaré-Birkhoff-Witt theorem and Cartier-Milnor-Moore theorem, valid for cocommutative bialgebras, to a large class of generalized bialgebras.

Technically we work in the theory of operads which allows us to state our main results and permits us to give it a conceptual proof. A generalized bialgebra type is determined by two operads : one for the coalgebra structure  $\mathcal{C}$ , and one for the algebra structure  $\mathcal{A}$ . There is also a compatibility relation relating the two. Under some conditions, the primitive part of such a generalized bialgebra is an algebra over some sub-operad of  $\mathcal{A}$ , denoted  $\mathcal{P}$ . The structure theorem gives conditions under which a connected generalized bialgebra is cofree (as a connected  $\mathcal{C}$ -coalgebra) and can be re-constructed out of its primitive part by means of an enveloping functor from  $\mathcal{P}$ -algebras to  $\mathcal{A}$ -algebras. The classical case is  $(\mathcal{C}, \mathcal{A}, \mathcal{P}) = (\text{Com}, \text{As}, \text{Lie})$ .

This structure theorem unifies several results, generalizing the PBW and the CMM theorems, scattered in the literature. We treat many explicit examples and suggest a few conjectures.

## **Résumé (Bigèbres généralisées et triples d'opérades)**

On introduit la notion de bigèbre généralisée, qui inclut la notion de bigèbre classique (algèbre de Hopf) et bien d'autres, comme, par exemple, l'algèbre tensorielle munie de la déconcaténation comme coproduit. On montre que, sous des hypothèses raisonnables, une bigèbre généralisée connexe est entièrement déterminée par sa partie

primitive. Ce théorème de structure étend à la fois le théorème classique de Poincaré-Birkhoff-Witt et le théorème de Cartier-Milnor-Moore valables pour les bigèbres co-commutatives, à une large classe de bigèbres généralisées.

On travaille dans le cadre de la théorie des opérades qui nous permet d'énoncer le résultat principal et d'en donner une démonstration conceptuelle. Un type de bigèbres généralisées est déterminé par deux opérades, l'une pour la structure de cogèbre, notée  $\mathcal{C}$ , l'autre pour la structure d'algèbre, notée  $\mathcal{A}$ . Ces deux structures sont reliées par certaines relations de compatibilité. Le théorème de structure donne des conditions sous lesquelles une bigèbre généralisée connexe est colibre (en tant que  $\mathcal{C}$ -cogèbre connexe) et peut être re-construite à partir de sa partie primitive grâce à un foncteur du type "algèbre enveloppante" des  $\mathcal{P}$ -algèbres dans les  $\mathcal{A}$ -algèbres. Le cas classique est  $(\mathcal{C}, \mathcal{A}, \mathcal{P}) = (Com, As, Lie)$ .

Ce théorème de structure unifie plusieurs généralisations du théorème PBW et du théorème CMM déjà présentes dans la littérature. On donne plusieurs exemples explicites et on formule quelques conjectures.

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<sup>(1)</sup> Added in proof: this problem has been settled in [55].



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