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Algebraic Study of Systems of Partial Differential Equations

(Master's Thesis, Tokyo University, December 1970)

Masaki KASHIWARA

Translated by Andrea D'AGNOLO and Jean-Pierre SCHNEIDERS

Abstract - This Mémoire is a translation of M. Kashiwara's thesis. In this pioneering work, the author initiates the study of systems of linear partial differential equations with analytic coefficients from the point of view of modules over the ring $\mathcal{D}$ of differential operators. Following some preliminaries on good filtrations and non-commutative localization, the author introduces the notion of characteristic variety and of multiplicity of a $\mathcal{D}$-module. Then he shows that the classical Cauchy-Kovalevskaya theorem may be generalized as a formula for the solutions of non-characteristic inverse images of $\mathcal{D}$-modules. Among the applications of this result, we find a solvability criterion in the complex domain and a study of the Cauchy problem for hyperfunctions. The author also investigates the homological properties of $\mathcal{D}$-modules linking, in particular, their homological dimension to the codimension of their characteristic variety. The thesis concludes with an index formula for holonomic systems on smooth complex curves.

Résumé - Ce mémoire est une traduction de la thèse de M. Kashiwara. Dans ce travail de pionnier, l'auteur entreprend l'étude des systèmes d'équations aux dérivées partielles linéaires à coefficients analytiques du point de vue des modules sur l'anneau $\mathcal{D}$ des opérateurs différentiels. Après quelques préliminaires sur les bonnes filtrations et la localisation non-commutative, l'auteur introduit la notion de variété caractéristique et de multiplicité d'un $\mathcal{D}$-module. Ensuite, il montre que le théorème classique de Cauchy-Kovalevskaya peut être généralisé en une formule pour les solutions des images inverses non-caractéristiques des $\mathcal{D}$-modules. Parmi les applications de ce résultat, nous trouvons un critère de résolubilité dans le domaine complexe et une étude du problème de Cauchy pour les hyperfonctions. L'auteur examine également les propriétés homologiques des $\mathcal{D}$-modules, reliant en particulier leur dimension homologique à la codimension de leur variété caractéristique. La thèse se conclut avec une formule d'indice pour les systèmes holonomes sur les courbes complexes lisses.

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To the memory of our friend Emmanuel Andronikof

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Translators' Foreword

The study of analytic linear partial differential equations using the powerful tools of homological algebra and sheaf theory began in the seventies. It has proved to be a very successful approach for a broad range of mathematical questions (microlocal analysis, index formulas, representation theory, etc.). The set of results obtained using these methods forms what is often called "Algebraic Analysis".

One of the important components of algebraic analysis is the study of analytic $\mathcal{D}$-modules (i.e., modules over the ring of linear partial differential operators with analytic coefficients). In his master's thesis (Tokyo University, December 1970), M. Kashiwara did a very important pioneering work on this subject. Until now, this text has been unavailable to a large public (it only appeared in a local handwritten publication in Japanese).

Although various expository texts on this subject are now available, we feel that Kashiwara's thesis is still interesting in its own right; not only as an important landmark in the historical development of algebraic analysis, but also as an illuminating introduction to analytic $\mathcal{D}$-modules.

In this volume, we present an almost faithful translation of this work. The only differences with the original consist in a few minor corrections and the use of up-to-date notations. We felt it unnecessary to mention these small changes explicitly.

We hope that in reading Kashiwara's thesis the reader's interest in our field of research will be stimulated. Of course, since 1970, the field has evolved much. To get a better perspective, one may refer to the list of suggested further readings which can be found hereafter. Although far from exhaustive, this list may nevertheless serve as a source of pointers to the extensive research literature.

We wish to thank Tae Morikawa for her help in preparing a typewritten Japanese version of Kashiwara's handwritten notes, and Kimberly De Haan for her attempts to clean up our franco-italian English.

Paris, February 1995

Andrea D'Agnolo — Jean-Pierre Schneiders

Further Readings

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Foreword

The idea of regarding a system of linear equations as a module over a ring is basic to algebraic geometry. However, it only appeared in the '70s for systems of partial differential equations with analytic coefficients, after pioneering talks by Sato in the '60s and Quillen thesis in '64. The two seminal papers on this subject are certainly Kashiwara's thesis (December '70) and Bernstein's papers of '71 and '72 published in "Functional Analysis" [3, 4]. Unfortunately, Kashiwara's thesis has never been translated nor published, and only exists in the form of handwritten mimeographed notes. Nevertheless, it has been distributed, inside and outside Japan, and some people have found the material to their inspiration at its reading.

However, the aim of this publication is by no means historical. As it will become evident to the reader, this thesis could have been written last week (modulo minor modifications): it contains a great deal of little known or even unknown results and could be used almost without any changes as a textbook for post-graduate courses.

Twenty-five years have passed since this thesis was written, and $\mathcal{D}$-module theory is now a basic tool in many branches of Mathematics: linear partial differential equations, harmonic analysis and representation theory, algebraic geometry, etc. Without being exhaustive, let us describe a few directions in which $\mathcal{D}$-modules techniques and Kashiwara's contribution have been decisive.

Before $\mathcal{D}$-modules were studied for themselves, mathematicians were interested in their solutions (distributions, hyperfunctions, etc.). In this perspective, the theory takes its full strength with the microlocal point of view, introduced by M. Sato in '69 and developed with Kawai and Kashiwara in [26]. This paper is at the origin of what is now called "microlocal analysis", and gave rise to enormous literature in the '80s.

Among $\mathcal{D}$-modules, there is a class of particular importance — holonomic $\mathcal{D}$-modules — which generalizes the notion of ordinary differential equations. In his 1975 paper [9], Kashiwara proved that if $\mathcal{M}$ is holonomic, and if one calls *Sol* the functor which to a $\mathcal{D}$-module associates the complex of its holomorphic solutions, then $Sol(\mathcal{M})$ is constructible, and is even perverse (although the theory of perverse sheaves didn't exist at the time). That same year (see [23], p. 287), he gave a precise statement for the Riemann-Hilbert correspondence. Then with Oshima [18] and Kawai [16], he developed the theory of regular holonomic $\mathcal{D}$-modules (a generalization in higher dimension of the Picard-Fuchs theory) and in 1980 he gave a proof