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SYMBOLIC EXTENSIONS IN INTERMEDIATE SMOOTHNESS ON SURFACES

BY David BURGUET

ABSTRACT. – We prove that $\mathscr{C}^r$ maps with $r > 1$ on a compact surface have symbolic extensions, i.e., topological extensions which are subshifts over a finite alphabet. More precisely we give a sharp upper bound on the so-called symbolic extension entropy, which is the infimum of the topological entropies of all the symbolic extensions. This answers positively a conjecture of S. Newhouse and T. Downarowicz in dimension two and improves a previous result of the author [9].

RÉSUMÉ. – Nous montrons que toute dynamique de classe $\mathscr{C}^r$ avec $r > 1$ sur une surface compacte admet une extension symbolique, i.e. une extension topologique qui est un sous-décalage à alphabet fini. Nous donnons plus précisément une borne (optimale) sur l'infimum de l'entropie topologique de toutes les extensions symboliques. Ceci répond positivement à une conjecture de S. Newhouse and T. Downarowicz en dimension deux et améliore un résultat précédent de l'auteur [9].

1. Introduction

By a dynamical system (X, T) we mean a continuous map T on a compact metrizable space X. One well studied class of dynamical systems are the symbolic ones, i.e., closed subsets Y of $\mathscr{A}^{\mathbb{Z}}$, with a finite alphabet $\mathscr{A}$, endowed with the shift S. Such a pair (Y, S) is also called a subshift. Given a dynamical system (X, T) one wonders if there exists a symbolic extension (Y, S) of (X, T), i.e., a subshift (Y, S) along a continuous surjective map $\pi : Y \to X$ such that $\pi \circ S = T \circ \pi$. We first observe that dynamical systems with symbolic extensions have necessarily finite topological entropy. When a dynamical system has symbolic extensions we are interested in minimizing their entropy. The topological symbolic extension entropy $h_{\mathrm{sex}}(T) = \inf\{h_{\mathrm{top}}(Y, S)\colon (Y, S) \text{ is a symbolic extension of } (X, T)\}$ estimates how the dynamical system (X, T) differs from a symbolic extension from the point of view of entropy. The problem of the existence of symbolic extensions leads to a deep theory of entropy which was developed mainly by M. Boyle and T. Downarowicz, who related the existence of symbolic extensions and their entropy with the convergence of the entropy of (X, T) computed at finer and finer scales [2].

By using a result of J. Buzzi [12] involving Yomdin's theory, M. Boyle, D. Fiebig and U. Fiebig [3] proved that $\mathscr{C}^\infty$ maps on a compact manifold admit principal symbolic extensions, i.e., symbolic extensions which preserve the entropy of invariant measures [3]. On the other hand $\mathscr{C}^1$ maps without symbolic extensions have been built in several works [20], [1], [7], [16], [13], [14]. In the present paper we consider dynamical systems of intermediate smoothness, i.e., $\mathscr{C}^r$ maps T on a compact manifold with $1 < r < +\infty$ (we mean that T admits a derivative or order $\lceil r-1\rceil$ which is $r-\lceil r-1\rceil$-Hölder). T. Downarowicz and A. Maass have recently proved that $\mathscr{C}^r$ maps of the interval $f : [0,1] \to [0,1]$ with $1 < r < +\infty$ have symbolic extensions [19]. More precisely they showed that $h_{\mathrm{sex}}(f) \leq h_{\mathrm{top}}(f) + \frac{\log \|f'\|_\infty}{r-1}$. The author built explicit examples [7] proving that this upper bound is sharp. Similar $\mathscr{C}^r$ examples with large symbolic extension entropy have been previously built by T. Downarowicz and S. Newhouse for diffeomorphisms in higher dimension [20]. The results of T. Downarowicz and A. Maass have been extended by the author in any dimension to nonuniformly entropy expanding maps (i.e., $\mathscr{C}^1$ maps whose ergodic measures with positive entropy have nonnegative Lyapunov exponents) of class $\mathscr{C}^r$ with $1 < r < +\infty$ [8]. More recently the author also proved the existence of symbolic extensions for $\mathscr{C}^2$ surface local diffeomorphisms [9]. T. Downarowicz and S. Newhouse have conjectured in [20] that $\mathscr{C}^r$ maps on a compact manifold with $r > 1$ have symbolic extensions. The following theorem answers affirmatively to this conjecture in dimension 2 and gives a sharp upper bound for the symbolic extension entropy in the case of diffeomorphisms. This extends thus the results of [9]. When $T : M \to M$ is a $\mathscr{C}^1$ map on a compact Riemannian manifold $(M, \|\,.\,\|)$ we denote by $R(T)$ the exponential growth of the derivative, i.e., $R(T) = \lim_{n\to+\infty} \frac{\log^+ \|DT^n\|}{n}$. This quantity does not depend on the choice of the Riemannian metric $\|\,.\,\|$ on M.

Theorem 1. – *Let $T : M \to M$ be a $\mathscr{C}^r$ map on a compact surface M with $r > 1$. Then T admits symbolic extensions and*

$$h_{\mathrm{sex}}(T) \leq h_{\mathrm{top}}(T) + \frac{4R(T)}{r-1}.$$

Moreover, if T is a local surface diffeomorphism, then

$$h_{\mathrm{sex}}(T) \leq h_{\mathrm{top}}(T) + \frac{R(T)}{r-1}.$$

The paper is organized as follows. We first recall the background of the theory of symbolic extensions and properties of continuity of the sum of the positive Lyapunov exponents. Following S. Newhouse we also recall how the local entropy is bounded from above by the local volume growth of smooth disks. Then we state our main results and as in [9] we reduce them to a Reparametrization Lemma of Bowen's balls in a similar (but finer) approach of the classical Yomdin theory. The last sections are devoted to the proof of the Reparametrization Lemma.

2. Preliminaries

In the following we denote by $\mathscr{M}(X,T)$ the set of invariant Borel probability measures of the dynamical system (X,T) and $\mathscr{M}_e(X,T)$ the subset of ergodic measures. We endow

$\mathcal{M}(X,T)$ with the weak star topology. Since X is a compact metric space, this topology is metrizable. We denote by dist a metric on $\mathcal{M}(X,T)$. It is well known that $\mathcal{M}(X,T)$ is compact and convex and its extreme points are exactly the ergodic measures. Moreover if $\mu \in \mathcal{M}(X,T)$ there exists a unique Borel probability measure M_μ on $\mathcal{M}(X,T)$ supported by $\mathcal{M}_e(X,T)$ such that for all Borel subsets B of X we have $\mu(B) = \int \nu(B)dM_\mu(\nu)$. This is the so called ergodic decomposition of μ. A bounded real Borel map $f : \mathcal{M}(X,T) \to \mathbb{R}$ is said to be harmonic if $f(\mu) = \int_{\mathcal{M}_e(X,T)} f(\nu)dM_\mu(\nu)$ for all $\mu \in \mathcal{M}(X,T)$. It is a well known fact that affine upper semicontinuous maps are harmonic. The measure theoretical entropy $h : \mathcal{M}(X,T) \to \mathbb{R}^+$ is always harmonic [30] but is not upper semicontinuous in general. It may not be upper semicontinuous even for $\mathcal{C}^r$ map for any $r \in \mathbb{R}^+$ [24]. However h is upper semicontinuous for $\mathcal{C}^\infty$ maps [27].

If f is a bounded real Borel map defined on $\mathcal{M}_e(X,T)$, the harmonic extension $\overline{f}$ of f is the function defined on $\mathcal{M}(X,T)$ by:

$$\overline{f}(\mu) := \int_{\mathcal{M}_e(X,T)} f(\nu)dM_\mu(\nu).$$

It is easily seen that $\overline{f}$ coincides with f on $\mathcal{M}_e(X,T)$ and that $\overline{f}$ is harmonic.

2.1. Entropy structure

The measure theoretical entropy function can be computed in many ways as limits of nondecreasing sequences of nonnegative functions defined on $\mathcal{M}(X,T)$ (with decreasing sequences of partitions, formula of Brin-Katok, ...). The entropy structures are such particular sequences whose convergence reflect the topological dynamic: they allow for example to compute the tail entropy [6] [17], but also especially the symbolic extension entropy [2] [17] (see below for precise statements).

We skip the formal definition of entropy structures, but we recall a basic fact. Two nondecreasing sequences, $(h_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$, of nonnegative functions defined on $\mathcal{M}(X,T)$ are said to be uniformly equivalent if for all $\gamma > 0$ and for all $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that $h_l > g_k - \gamma$ and $g_l > h_k - \gamma$. Two entropy structures are uniformly equivalent and any nondecreasing sequence of nonnegative functions which is uniformly equivalent to an entropy structure is itself an entropy structure. In other terms the set of entropy structures is an equivalence class for the above relation.

We recall now Lemma 1 of [9] which relates the entropy structures of a given dynamical system with those of its inverse and powers.

LEMMA 1. – *Let (X,T) be a dynamical system with finite topological entropy and let $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ be an entropy structure of T^p with $p \in \mathbb{N}\setminus\{0\}$ (when T is a homeomorphism we consider $p \in \mathbb{Z}\setminus\{0\}$). Then the sequence $\frac{1}{|p|}\mathcal{H}|_{\mathcal{M}(X,T)} = \left(\frac{h_k|_{\mathcal{M}(X,T)}}{|p|}\right)_{k\in\mathbb{N}}$ defines an entropy structure of T.*

We finally check that the minimum of two entropy structures defines again an entropy structure.

LEMMA 2. – *Let (X,T) be a dynamical system with finite topological entropy. If $\mathcal{H} = (h_k)_k$ and $\mathcal{G} = (g_k)_k$ are two entropy structures, then $\min(\mathcal{H},\mathcal{G}) := (\min(h_k, g_k))_k$ is also an entropy structure.*

Proof. – Let $\gamma > 0$ and $k \in \mathbb{N}$. As $\mathscr{H}$ and $\mathscr{G}$ are both entropy structures, they are in particular uniformly equivalent. Therefore there exists an integer l such that $h_l > g_k - \gamma$ and $g_l > h_k - \gamma$. We can assume that $l > k$ so that $h_l \geq h_k$ by monotonicity of $\mathscr{H}$. Therefore $h_l > \min(h_k, g_k) - \gamma$ and $\min(h_l, g_l) > h_k - \gamma$. □

2.2. Tail entropy

In order to study the properties of upper semicontinuity of the entropy function of a dynamical system and in particular the existence of measures of maximal entropy, M. Misiurewicz introduced in the seventies the following quantity which is now known as the tail entropy of the system. Let us first recall some usual notions relating to the entropy of dynamical systems (we refer to [30] for a general introduction to entropy).

Consider a continuous map $T : X \to X$ with (X, d) a compact metric space. Let $n \in \mathbb{N}$ and $\delta > 0$. A subset F of X is called a (n, δ) separated set when for all $x, y \in F$ there exists $0 \leq k < n$ such that $d(T^k x, T^k y) \geq \delta$. Let Y be a subset of X. A subset F of Y is called a (n, δ) spanning set of Y when for all $y \in Y$ there exists $z \in F$ such that $d(T^k y, T^k z) < \delta$ for all $0 \leq k < n$. Given a point $x \in X$ we denote by $B(x, n, \delta)$ the Bowen's ball centered at x of radius δ and length n:

$$B(x, n, \delta) := \{y \in X,\ d(T^k y, T^k x) < \delta \text{ for } k = 0, \ldots, n-1\}.$$

The tail entropy, $h^*(T)$, of (X, T) is then defined by

$$h^*(T) := \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \sup_{x \in X} \frac{1}{n} \log \min \{\sharp F \ \ F \text{ is a } (n, \delta) \text{ spanning set of } B(x, n, \epsilon)\}.$$

This quantity is a topological invariant which estimates the entropy appearing at arbitrarily small scales. The tail entropy bounds from above the defect of upper semicontinuity of the entropy function [25]:

$$\forall \mu \in \mathscr{M}(X, T),\ \limsup_{\nu \to \mu} h(\nu) - h(\mu) \leq h^*(T).$$

In general the supremum of the defect of upper semicontinuity of the entropy function differs from the tail entropy. But it is easily seen that for any entropy structure $(h_k)_k$ of (X, T), we have $\limsup_{\nu \to \mu} h(\nu) - h(\mu) \leq \lim_k \limsup_{\nu \to \mu} (h - h_k)(\nu)$ and T. Downarowicz proved then the following variational principle [17] (see also [6]):

$$\sup_{\mu \in \mathscr{M}(X,T)} \lim_k \limsup_{\nu \to \mu} (h - h_k)(\nu) = \lim_k \sup_{\mu \in \mathscr{M}(X,T)} (h - h_k)(\mu) = h^*(T). \tag{1}$$

By using Yomdin's theory J. Buzzi [12] established the following upper bound on the tail entropy of $\mathscr{C}^r$ maps T on a compact manifold M with $r > 1$:

$$h^*(T) \leq \frac{\dim(M)}{r} R(T). \tag{2}$$

This inequality is known to be sharp for noninvertible maps [12], [29]. We will prove in the present paper a similar sharp upper bound on the tail entropy of $\mathscr{C}^r$ surface diffeomorphisms with $r > 1$ (see Theorem 5 below).

When $h^*(T) = 0$ the dynamical system (X, T) is said to be asymptotically h-expansive. For example, uniformly hyperbolic dynamical systems or piecewise monotone interval maps are asymptotically h-expansive. Then entropy structures are converging uniformly to the