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SHIMURA VARIETIES WITH $\Gamma_1(p)$-LEVEL VIA HECKE ALGEBRA ISOMORPHISMS: THE DRINFELD CASE

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ABSTRACT. – We study the local factor at p of the semi-simple zeta function of a Shimura variety of Drinfeld type for a level structure given at p by the pro-unipotent radical of an Iwahori subgroup. Our method is an adaptation to this case of the Langlands-Kottwitz counting method. We explicitly determine the corresponding test functions in suitable Hecke algebras, and show their centrality by determining their images under the Hecke algebra isomorphisms of Goldstein, Morris, and Roche.

RÉSUMÉ. – On étudie le facteur local en p de la fonction zêta semi-simple d'une variété de Shimura du type de Drinfeld, où la structure de niveau en p est donnée par le radical pro-unipotent d'un sous-groupe d'Iwahori. La méthode suivie est une adaptation à ce cas de la méthode de comptage de Langlands-Kottwitz. On détermine de façon explicite la fonction test dans l'algèbre de Hecke correspondante ; puis on démontre que c'est un élément central en déterminant ses images sous des isomorphismes d'algèbres de Hecke dus à Goldstein, Morris et Roche.

1. Introduction

Many authors have studied the Hasse-Weil zeta functions of Shimura varieties, and more generally the relations between the cohomology of Shimura varieties and automorphic forms. In the approach of Langlands and Kottwitz, an important step is to express the semi-simple Lefschetz number in terms of a sum of orbital integrals that resembles the geometric side of the Arthur-Selberg trace formula. The geometry of the reduction modulo $\mathfrak{p}$ of the Shimura variety determines which functions appear as test functions in the orbital integrals. Via the Arthur-Selberg trace formula, the traces of these test functions on automorphic representations intervene in the expression of the semi-simple Hasse-Weil zeta function in terms of semi-simple automorphic L-functions. Thus, at the heart of this method is the

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precise determination of the test functions themselves, for any given level structure, and this precise determination is a problem that seems of interest in its own right.

Let us describe more precisely what we mean by test functions. Let $Sh_K = Sh(G, h^{-1}, K)$ denote the Shimura variety attached to a connected reductive $\mathbb{Q}$-group G, a family of Hodge-structures h, and a compact open subgroup $K \subset G(\mathbb{A}_f)$. We fix a prime number p and assume that K factorizes as $K = K^p K_p \subset G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p)$. Let $E \subset \mathbb{C}$ denote the reflex field and suppose that for a fixed prime ideal $\mathfrak{p} \subset \mathcal{O}_E$ over p, Sh_K possesses a suitable integral model, still denoted by Sh_K, over the ring of integers $\mathcal{O}_{E_\mathfrak{p}}$ in the $\mathfrak{p}$-adic completion $E_\mathfrak{p}$. Also, for simplicity let us assume that $E_\mathfrak{p} = \mathbb{Q}_p$ (this will be the case in the body of this paper). Write $R\Psi = R\Psi^{Sh_K}(\bar{\mathbb{Q}}_\ell)$ for the complex of nearby cycles on the special fiber of Sh_K, where $\ell \neq p$ is a fixed prime number. For $r \geq 1$, let k_r denote an extension of degree r of $\mathbb{F}_p$, and consider the semi-simple Lefschetz number defined as the element in $\bar{\mathbb{Q}}_\ell$ given by the sum

$$\mathrm{Lef}^{ss}(\Phi_\mathfrak{p}^r, R\Psi) = \sum_{x \in Sh_K(k_r)} \mathrm{Tr}^{ss}(\Phi_\mathfrak{p}^r, R\Psi_x),$$

where $\Phi_\mathfrak{p}$ denotes the Frobenius morphism for $Sh_K \otimes_{\mathcal{O}_{E_\mathfrak{p}}} \mathbb{F}_p$. (We refer the reader to [22] for a discussion of semi-simple trace on $\bar{\mathbb{Q}}_\ell$-complexes being used here.)

The group-theoretic description of $\mathrm{Lef}^{ss}(\Phi_\mathfrak{p}^r, R\Psi)$ emphasized by Langlands [43] and Kottwitz [33, 35] should take the form of a "counting points formula"

$$\mathrm{Lef}^{ss}(\Phi_\mathfrak{p}^r, R\Psi) = \sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0;\gamma,\delta)\, \mathrm{O}_\gamma(f^p)\, \mathrm{TO}_{\delta\sigma}(\phi_r).$$

The sum is over certain equivalence classes of triples $(\gamma_0;\gamma,\delta) \in G(\mathbb{Q}) \times G(\mathbb{A}_f^p) \times G(L_r)$, where L_r is the fraction field of the ring of Witt vectors $W(k_r)$, and where σ denotes the Frobenius generator of $\mathrm{Gal}(L_r/\mathbb{Q}_p)$. We refer to Section 8 for a discussion of the implicit measures and the remaining notation in this formula.

We take the open compact subgroup K_p of the kind that for any $r \geq 1$ it also defines an open compact subgroup K_{p^r} of $G(L_r)$. In terms of the Haar measure $dx_{K_{p^r}}$ on $G(L_r)$ giving K_{p^r} volume 1, we say $\phi_r = \phi_r(R\Psi)$ is "the" *test function* if the counting points formula above is valid. It should be an element in the Hecke algebra $\mathcal{H}(G(L_r), K_{p^r})$ of compactly-supported K_{p^r}-bi-invariant $\bar{\mathbb{Q}}_\ell$-valued functions on $G(L_r)$. The function ϕ_r is not uniquely determined by the formula, but a striking empirical fact is that a very nice choice for ϕ_r always seems to exist, namely, *we may always find a test function ϕ_r belonging to the center $\mathcal{Z}(G(L_r), K_{p^r})$ of $\mathcal{H}(G(L_r), K_{p^r})$*. (A very precise conjecture can be formulated using the *stable Bernstein center* (cf. [59]), but this will be done on another occasion by one of us (T. H.).)

In fact, test functions seem to be given in terms of the Shimura cocharacter μ_h by a simple rule. Let us illustrate this with the known examples. Assume $G_{\mathbb{Q}_p}$ is unramified. Let $\mu = \mu_h$ denote the minuscule cocharacter of $G_{\mathbb{Q}_p}$ associated to h and the choice of an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ extending $\mathfrak{p}$, and let μ^* denote its dual (i.e. $\mu^* = -w_0(\mu)$ for the longest element w_0 of the relative Weyl group). If K_p is a *hyperspecial* maximal compact subgroup, and Sh_K is PEL of type A or C, then Kottwitz [37] shows that Sh_K has an integral model with good reduction at $\mathfrak{p}$, and that ϕ_r may be taken to be the characteristic function $1_{K_{p^r}\mu^*(p)K_{p^r}}$ in the (commutative) spherical Hecke algebra $\mathcal{H}(G(L_r), K_{p^r})$. Next suppose that Sh_K is of PEL type, that $G_{\mathbb{Q}_p}$ is split of type A or C, and that K_p is a parahoric subgroup. Then in [22], [18] it is proved that ϕ_r may be taken to be the *Kottwitz function* $k_{\mu^*,r} = p^{r\dim(Sh_K)/2}\, z_{\mu^*,r}^{K_{p^r}}$ in $\mathcal{Z}(G(L_r), K_{p^r})$, thus confirming a conjecture of Kottwitz. Here for an Iwahori subgroup I_r contained in K_{p^r} the symbol $z_{\mu^*,r}$ denotes the *Bernstein function* in $\mathcal{Z}(G(L_r), I_r)$ described in the appendix, and $z_{\mu^*,r}^{K_{p^r}}$ denotes the image of $z_{\mu^*,r}$ under the canonical isomorphism $\mathcal{Z}(G(L_r), I_r) \xrightarrow{\sim} \mathcal{Z}(G(L_r), K_{p^r})$ given by convolution with the characteristic function of K_{p^r} (see [19]).

In this article, we study Shimura varieties in the Drinfeld case. From now on we fix the associated "unitary" group G, coming from an involution on a semi-simple algebra D with center an imaginary quadratic extension $F/\mathbb{Q}$ and with $\dim_F D = d^2$ (see Section 3). We make assumptions guaranteeing that $G_{\mathbb{Q}_p} = \mathrm{GL}_d \times \mathbb{G}_m$, so that we may identify μ with the Drinfeld type cocharacter $\mu_0 = (1, 0^{d-1})$ of GL_d. For K_p we take either the Iwahori subgroup I of $\mathrm{GL}_d(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ of elements where the matrices in the first factor are congruent modulo p to an upper triangular matrix, or its pro-unipotent radical I^+ where the matrices in the first factor are congruent modulo p to a unipotent upper triangular matrix. The first case leads to the $\Gamma_0(p)$-level moduli scheme, and the second to the $\Gamma_1(p)$-level moduli scheme.

The $\Gamma_0(p)$-level moduli scheme $\mathcal{A}_0$ is defined as parametrizing a chain of polarized abelian varieties of dimension d^2 with an action of a maximal order in D. The moduli problem $\mathcal{A}_1$ is defined using the Oort-Tate theory [58] of group schemes of order p. Namely, $\mathcal{A}_1 \to \mathcal{A}_0$ is given by choosing an Oort-Tate generator for each group scheme G_i $(i = 1, \ldots, d)$ coming from the flag (3.3.5) of p-divisible groups attached to a point in $\mathcal{A}_0$. We use the geometry of the canonical morphism

$$\pi : \mathcal{A}_1 \to \mathcal{A}_0$$

to study the nearby cycles $R\Psi_1$ on $\mathcal{A}_1 \otimes \mathbb{F}_p$. The following theorem summarizes Propositions 12.1.1 and 12.2.1. It is the analogue for $\mathcal{A}_1$ of the theorem for $\mathcal{A}_0$ mentioned above, which identifies the test function $\phi_r(R\Psi_0)$ with the Kottwitz function $k_{\mu^*,r}$ in the center of the Iwahori Hecke algebra.

Theorem 1.0.1. – *Consider the Shimura variety $Sh_K = \mathcal{A}_{1,K^p}$ in the Drinfeld case, where $K_p = I^+$ is the pro-unipotent radical of an Iwahori subgroup $I \subset G(\mathbb{Q}_p)$. Let $I_r^+ \subset I_r$ denote the corresponding subgroups of $G_r := G(L_r)$. Then with respect to the Haar measure on G_r giving I_r^+ volume 1, the test function $\phi_{r,1} = \phi_r(R\Psi_1)$ is an explicit function belonging to the center $\mathcal{Z}(G_r, I_r^+)$ of $\mathcal{H}(G_r, I_r^+)$.*

(See Section 12 for the spectral characterization of $\phi_{r,1}$ and an explicit formula for it.)

The centrality of test functions facilitates the pseudo-stabilization of the counting points formula. The test function $\phi_{r,1} \in \mathcal{Z}(G_r, I_r^+)$ is characterized by its traces on depth-zero principal series representations, and a similar statement applies to its image under an appropriate base-change homomorphism

$$b_r : \mathcal{Z}(G_r, I_r^+) \to \mathcal{Z}(G, I^+)$$

which we discuss in Section 10. As for the spherical case [5, 41] and the parahoric case [19], the present b_r yields pairs of associated functions with matching (twisted) stable orbital integrals [20] (see Theorem 10.2.1 for a statement), and this plays a key role in the pseudo-stabilization, cf. Subsection 13.4. In our case, the image $b_r(\phi_{r,1})$ of $\phi_{r,1}$ has the following nice spectral expression (Cor. 13.2.2).

Proposition 1.0.2. – *Write $f_{r,1} := b_r(\phi_{r,1}) \in \mathcal{Z}(G, I^+)$. Let (r_{μ^*}, V_{μ^*}) denote the irreducible representation with extreme weight μ^* of the L-group ${}^L(G_{\mathbb{Q}_p})$. Let $\Phi \in W_{\mathbb{Q}_p}$ denote a geometric Frobenius element and let $I_p \subset W_{\mathbb{Q}_p}$ denote the inertia subgroup. Then for any irreducible smooth representation π_p of $G(\mathbb{Q}_p)$ with Langlands parameter $\varphi'_{\pi_p} : W'_{\mathbb{Q}_p} \to \widehat{G}$, we have*

$$\operatorname{tr} \pi_p(f_{r,1}) = \dim(\pi_p^{I^+})\, p^{r\langle \rho, \mu^* \rangle} \operatorname{Tr}(r_{\mu^*} \circ \varphi'_{\pi_p}(\Phi^r),\ V_{\mu^*}^{I_p}).$$

It is expected (cf. [50]) that semi-simple zeta functions can be expressed in terms of semi-simple automorphic L-functions. By their very definition, the semi-simple local L-factors $L^{ss}(s, \pi_p, r_{\mu^*})$ involve expressions as on the right hand side above. In Section 13.4 we use this and Theorem 10.2.1 to describe the semi-simple local factor $Z^{ss}_{\mathfrak{p}}(s, \mathcal{A}_{1,K^p})$ of $\mathcal{A}_1 = \mathcal{A}_{1,K^p}$, when the situation is also "simple" in the sense of Kottwitz [36].

Theorem 1.0.3. – *Suppose D is a division algebra, so that the Shimura variety $\mathcal{A}_1 = \mathcal{A}_{1,K^p}$ is proper over $\operatorname{Spec} \mathbb{Z}_p$ and has "no endoscopy" (cf. [36]). Let $n = d - 1$ denote the relative dimension of $\mathcal{A}_{1,K^p}$. For every integer $r \geq 1$, the alternating sum of the semi-simple traces*

$$\sum_{i=0}^{2n} (-1)^i \operatorname{Tr}^{ss}(\Phi^r_{\mathfrak{p}}\,,\ \mathrm{H}^i(\mathcal{A}_{1,K^p} \otimes_E \bar{E}_{\mathfrak{p}}, \bar{\mathbb{Q}}_\ell))$$

equals the trace

$$\operatorname{Tr}(1_{K^p} \otimes f_{r,1} \otimes f_\infty\,,\ \mathrm{L}^2(G(\mathbb{Q})\, A_G(\mathbb{R})^\circ \backslash G(\mathbb{A}))).$$

Here A_G is the split component of the center of G and f_∞ is the function at the Archimedean place defined by Kottwitz [36].

In Section 13.4 we derive from the two previous facts the following result.

Corollary 1.0.4. – *In the situation above, we have*

$$Z^{ss}_{\mathfrak{p}}(s, \mathcal{A}_{1,K^p}) = \prod_{\pi_f} L^{ss}(s - \frac{n}{2}, \pi_p, r_{\mu^*})^{a(\pi_f)\dim(\pi_f^K)},$$

where the product runs over all admissible representations π_f of $G(\mathbb{A}_f)$. The integer $a(\pi_f)$ is given by

$$a(\pi_f) = \sum_{\pi_\infty \in \Pi_\infty} m(\pi_f \otimes \pi_\infty) \operatorname{tr} \pi_\infty(f_\infty),$$

where $m(\pi_f \otimes \pi_\infty)$ is the multiplicity of $\pi_f \otimes \pi_\infty$ in $\mathrm{L}^2(G(\mathbb{Q}) A_G(\mathbb{R})^0 \backslash G(\mathbb{A}))$. Here Π_∞ is the set of admissible representations of $G(\mathbb{R})$ having trivial central character and trivial infinitesimal character.

We refer to Kottwitz's paper [36] for further discussion of the integer $a(\pi_f)$. Note that in the above product there are only a finite number of representations π_f of $G(\mathbb{A}_f)$ such that $a(\pi_f)\dim(\pi_f^K) \neq 0$.

Of course, this corollary is in principle a special case of the results of Harris and Taylor in [25] which are valid for any level structure. However, our emphasis here is on the explicit determination of test functions and their structure. To be more precise, let us return to the general Drinfeld case, where D is not necessarily a division algebra. By the compatibility of