ISSN 0012-9593

quatrième série - tome 52 fascicule 5 septembre-octobre 2019

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Tomoyuki ABE & Hélène ESNAULT

A Lefschetz theorem for overconvergent isocrystals with Frobenius structure

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 52, 2019, p. 1243 à 1264

A LEFSCHETZ THEOREM FOR OVERCONVERGENT ISOCRYSTALS WITH FROBENIUS STRUCTURE

BY TOMOYUKI ABE AND HÉLÈNE ESNAULT

ABSTRACT. – We show a Lefschetz theorem for irreducible overconvergent F-isocrystals on smooth varieties defined over a finite field. We derive several consequences from it.

RÉSUMÉ. – Nous montrons un théorème de Lefschetz pour les F-isocristaux surconvergents sur des variétés lisses définies sur un corps fini. Nous en tirons plusieurs conséquences.

Introduction

Let X_0 be a normal geometrically connected scheme of finite type defined over a finite field $\mathbb{F}_q$, let $\mathcal{F}_0$ be an irreducible lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf with finite determinant (thus in fact $\mathcal{F}_0$ is an étale sheaf as well), where $\ell \neq p = \mathrm{char}(\mathbb{F}_q)$. In Weil II [8, Conj. 1.2.10], Deligne conjectured the following.

(i) The sheaf $\mathcal{F}_0$ is of weight 0.

(ii) There is a number field $E \subset \overline{\mathbb{Q}}_\ell$ such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, the characteristic polynomial $f_x(\mathcal{F}_0, t) := \det(1 - tF_x \mid \mathcal{F}_{0,\bar{x}})$ lies in $E[t]$, where F_x is the geometric Frobenius of x.

(iii) For any $\ell' \neq p$ and any embedding $\sigma\colon E \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$, for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, any root of $\sigma f_x(\mathcal{F}_0, t)$ is an ℓ'-adic unit.

(iv) For any σ as in (iii), there is an irreducible $\overline{\mathbb{Q}}_{\ell'}$-lisse sheaf $\mathcal{F}_{0,\sigma}$, called the σ-companion, such that $\sigma f_x(\mathcal{F}_0, t) = f_x(\mathcal{F}_{0,\sigma}, t)$.

(v) There is a crystalline version of (iv).

The first author is supported by Grant-in-Aid for Young Scientists (A) 16H05993. The second author is supported by the Einstein program.

 doi:10.24033/asens.2408

Deligne's conjectures (i)–(iv) have been proved by Lafforgue [18, Thm. VII.6] when X_0 is a smooth curve, as a corollary of the Langlands correspondence, which is proven showing that automorphic forms are in some sense motivic.

When X_0 has dimension at least 2, the automorphic side on which one could rely to prove Deligne's conjectures is not available: there is no theory of automorphic forms in higher dimension. The problem then becomes how to reduce, by geometry, the statements to dimension 1. For (i) and (iii), one proves a Lefschetz theorem (see [12, Thm. 2.15], [10, 1.5–1.9], [13, B1]):

0.1. Theorem. – *On X_0 smooth, for any closed point x_0, there exists a smooth curve C_0 and a morphism $C_0 \to X_0$ such that $x_0 \to X_0$ lifts to $x_0 \to C_0$, and such that the restriction of $\mathcal{V}_0$ to C_0 remains irreducible.*

Using Theorem 0.1, Deligne proved (ii) ([10, Thm. 3.1]), and Drinfeld, using (ii), proved (iv) in ([12, Thm. 1.1]), assuming in addition X_0 to be smooth. In particular Drinfeld proved in [12, Thm. 2.5] the following key theorem.

0.2. Theorem. – *If X_0 is smooth, given a number field $E \subset \overline{\mathbb{Q}}_\ell$, and a place λ of E dividing ℓ, a collection of polynomials $f_x(t) \in E[t]$ indexed by any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, such that the following two conditions are satisfied:*

(i) *for any smooth curve C_0 with a morphism $C_0 \to X_0$ and any $n > 0$ and $x \in C_0(\mathbb{F}_{q^n})$, there exists a lisse étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{V}_0^{C_0}$ on C_0 with monodromy in $\mathrm{GL}(r, E_\lambda)$ such that $f_x(t) = f_x(\mathcal{V}_0^{C_0}, t)$, where E_λ is the completion of E with respect to the place λ;*

(ii) *there exists a finite étale cover $X'_0 \to X_0$ such that $\mathcal{V}_0^{C_0}$ is tame on all C_0 factoring through $X'_0 \to X_0$.*

Then there exists a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{V}_0$ on X_0 with monodromy in $\mathrm{GL}(r, E_\lambda)$, such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, $f_x(t) = f_x(\mathcal{V}_0, t) \in E[t]$.

Further, to realize the assumptions of Theorem 0.2 in order to show the existence of $\mathcal{V}_{0,\sigma}$, Drinfeld uses Theorem 0.1 in [12, 4.1]. He constructs step by step the residual representations with monodromy in $\mathrm{GL}(r, \mathcal{O}_{E_\lambda}/\mathfrak{m}^n)$ for n growing, where $\mathcal{O}_{E_\lambda}$ is the ring of integers of E_λ and $\mathfrak{m}$ is its maximal ideal.

The formulation of (v) has been made explicit by Crew [7, 4.13]. The conjecture is that the crystalline category analogous to the category of Weil $\overline{\mathbb{Q}}_\ell$-sheaves is the category of overconvergent F-isocrystals (see Section 1.1 for the definitions). In order to emphasize the analogy between ℓ and p, one slightly reformulates the definition of companions. One replaces σ in (iii) by an isomorphism $\sigma\colon \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_{\ell'}$ (see [13, Thm. 4.4]), and keeps (iv) as it is. Here ℓ, ℓ' are any two prime numbers. For $\ell' = p, \ell \neq p$, and $\mathcal{V}$ an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf, one requests the existence of an overconvergent F-isocrystal M_0 on X_0 with eigenpolynomial $f_x(M_0, t)$ such that $f_x(M_0, t) = \sigma f_x(\mathcal{V}, t) \in \sigma(E)[t]$ for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, where $f_x(\mathcal{V}, t)$ is the characteristic polynomial of the geometric Frobenius at x on $\mathcal{V}$ (see Section 1.4 for the definitions). The isocrystal M_0 is called a *σ-companion to $\mathcal{V}$*. Given an irreducible overconvergent F-isocrystal M_0 with finite determinant on X_0, and σ as above, a lisse ℓ-adic Weil sheaf $\mathcal{V}$ on X_0 is a σ^{-1}-companion if $\sigma^{-1} f_x(M_0, t) = f_x(\mathcal{V}, t) \in E[t]$

at $x \in X_0(\mathbb{F}_{q^n})$ (see Definition 1.4). Similarly we can assume $p = \ell = \ell'$. This way we can talk on ℓ-adic or p-adic companions of either an M_0 or an $\mathscr{F}$. The companion correspondence should preserve the notions of irreducibility, finiteness of the determinant, the eigenpolynomials at closed points of X_0, and the ramification.

The conjecture in the strong form has been proven by the first author when X_0 is a smooth curve ([1, Intro. Thm.]). The aim of this article is to prove the following analog of Theorem 0.1 on X smooth.

0.3. Theorem (Theorem 3.10). – *Let X_0 be a smooth geometrically connected scheme over $\mathbb{F}_q$. Let M_0 be an irreducible overconvergent F-isocrystal with finite determinant. Then for every closed point $x_0 \to X_0$, there exists a smooth irreducible curve C_0 defined over k, together with a morphism $C_0 \to X_0$ and a factorization $x_0 \to C_0 \to X_0$, such that the pull-back of M_0 to C_0 is irreducible.*

Theorem 0.3, together with [10, Rmk. 3.10], footnote 2, and [1, Thm. 4.2.2] enable one to conclude that there is a number field $E \subset \overline{\mathbb{Q}}_p$ such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, $f_x(M_0, t) \in E[t]$ (see Lemma 4.1). This yields the p-adic analog of (i) over a smooth variety X_0. (See Section 4.6 when X_0 is normal). Then Theorem 0.2 implies the existence of ℓ-adic companions to a given irreducible overconvergent F-isocrystal M_0 with finite determinant (see Theorem 4.2). We point out that the existence of ℓ-adic companions has already been proven by Kedlaya in [17, Thm. 5.3] in a different way, using weights (see [17, §4, Intro.]), however not their irreducibility. The Lefschetz Theorem 3.10 implies that the companion correspondence preserves irreducibility.

Theorem 0.3 has other consequences (see Section 4), aside of the existence already mentioned of ℓ-adic companions. Deligne's finiteness theorem [13, Thm. 1.1] transposes to the crystalline side (see Corollary 4.3): on X_0 smooth, there are finitely many isomorphism classes of irreducible overconvergent F-isocrystals in bounded rank and bounded ramification, up to twist by a character of the finite field. One can also kill the ramification of an F-overconvergent isocrystal by a finite étale cover in Kedlaya's semistability reduction theorem (Remark 4.4).

We now explain the method of proof of Theorem 0.3. We replace M_0 by the full Tannakian subcategory $\langle M \rangle$ of the category of overconvergent F-isocrystals spanned by M over the algebraic closure $\overline{\mathbb{F}}_q$ (we drop the lower indices $_0$ to indicate this, see Section 1.1 for the definitions). We slightly improve the theorem ([11, Prop. 2.21 (a), Rmk. 2.29]) describing the surjectivity of an homomorphism of Tannaka groups in categorical terms in Lemma 1.6: the restriction functor $\langle M \rangle \to \langle M|_C \rangle$ to a curve $C \to X$ is an equivalence when it is fully faithful and any F-overconvergent isocrystal of rank 1 on C is torsion. Class field theory for F-overconvergent isocrystals ([2, Lem. 6.1] [(1)]) implies the torsion property. As for full faithfulness, the problem is of cohomological nature, one has to compute that the restriction homomorphism $H^0(X, N) \to H^0(C, N|_C)$ is an isomorphism for all objects N in $\langle M \rangle$. In the tame case, this is performed in Section 2 using the techniques developed in [3]. As a corollary, ℓ-adic companions exist in the tame case (see Proposition 2.8). In the wild case, Kedlaya's

(1) See Remark 4.6 for a correction of a mistake in this lemma.