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ASYMPTOTIC GEOMETRY OF NEGATIVELY CURVED MANIFOLDS OF FINITE VOLUME

BY Françoise DAL'BO, Marc PEIGNÉ, Jean-Claude PICAUD AND Andrea SAMBUSETTI

ABSTRACT. – We study the asymptotic behavior of simply connected Riemannian manifolds X of strictly negative curvature admitting a non-uniform lattice Γ. If the quotient manifold $\bar{X} = \Gamma\backslash X$ is asymptotically $1/4$-pinched, we prove that Γ is divergent and $U\bar{X}$ has finite Bowen-Margulis measure (which is then ergodic and totally conservative with respect to the geodesic flow); moreover, we show that, in this case, the volume growth of balls $B(x, R)$ in X is asymptotically equivalent to a purely exponential function $c(x)e^{\delta R}$, where δ is the topological entropy of the geodesic flow of $\bar{X}$. This generalizes Margulis' celebrated theorem to negatively curved spaces of finite volume. In contrast, we exhibit examples of lattices Γ in negatively curved spaces X (not asymptotically $1/4$-pinched) where, depending on the critical exponent of the parabolic subgroups and on the finiteness of the Bowen-Margulis measure, the growth function is exponential, lower-exponential or even upper-exponential.

RÉSUMÉ. – Nous décrivons le comportement asymptotique de variétés Riemanniennes simplement connexes X à courbure strictement négative, dont le groupe d'isométries contient des sous-groupes discrets Γ de co-volume fini. Plus précisément, nous montrons que lorsque la courbure est asymptotiquement 1/4-pincée, le groupe Γ est alors divergent et la mesure de Bowen-Margulis associée est finie; de plus, le volume des boules $B(x, R)$ de X est asymptotiquement équivalent à la fonction $c(x)e^{\delta R}$, où δ désigne l'exposant de Poincaré de Γ. Ce résultat généralise le célèbre théorème de Margulis au cas des réseaux non-uniformes. Nous construisons aussi toute une série d'exemples de variétés X à courbure strictement négative mais non asymptotiquement 1/4-pincée, pour lesquels le volume des boules $B(x, R)$ de X ne croît pas toujours de façon purement exponentielle.

1. Introduction

Let X be a complete, simply connected manifold with strictly negative curvature. In the sixties, G. Margulis [18], using measure theory on the foliations of the Anosov system defined by the geodesic flow, showed that if Γ is a uniform lattice of X (i.e., a torsionless, discrete

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group of isometries such that $\bar{X} = \Gamma\backslash X$ is compact), then the *orbital function* of Γ is asymptotically equivalent [(1)] to a purely exponential function:

$$v_\Gamma(x, y, R) = \#\{\gamma \in \Gamma \mid d(x, \gamma y) < R\} \sim c_\Gamma(x, y)e^{\delta(\Gamma)R},$$

where $\delta(\Gamma) = \lim_{R\to\infty} R^{-1} \ln v_\gamma(x, x, R)$ is the *critical exponent* of Γ. By integration over fundamental domains, one then obtains an asymptotic equivalence for the *volume growth function* of X:

$$v_X(x, R) = \mathrm{vol}B(x, R) \sim m(x)e^{\delta(\Gamma)R}.$$

It is well-known that the exponent $\delta(\Gamma)$ equals the *topological entropy* of the geodesic flow of $\bar{X}$ (see [19]) and that, for uniform lattices, it is the same as the *volume entropy* $\omega(X) = \limsup \frac{1}{R} \ln v_X(x, R)$ of the manifold X. The function $m(x)$, depending on the center of the ball, is the *Margulis function* of X.

Since then, this result has been generalized in different directions. Notably, G. Knieper showed in [17] that the volume growth function of a Hadamard space X (a complete, simply connected manifold with nonpositive curvature) of rank one admitting uniform lattices is *purely exponential* [(2)], that is

$$v_X(x, R) \asymp e^{\omega(X)R}.$$

In general, he showed that $v_X(x, R) \asymp R^{\frac{d-1}{2}} e^{\omega(X)R}$ for rank d manifolds; however, as far as the authors are aware, it is still unknown whether there exists a Margulis function for Hadamard manifolds of rank 1 with uniform lattices, i.e., a function $m(x)$ such that $v_X(x, R) \sim m(x)e^{\omega(X)R}$, even in the case of surfaces. Another remarkable case is that of *asymptotically harmonic manifolds* of strictly negative curvature, where the strong asymptotic homogeneity implies the existence of a Margulis function, even without compact quotients, cp. [5].

In another direction, it seems natural to ask what happens for a Hadamard space X of negative curvature admitting *nonuniform lattices* Γ (i.e., $\mathrm{vol}(\Gamma\backslash X) < \infty$): *is v_X purely exponential and, more precisely, does X admit a Margulis function?* Let us emphasize that if X also admits a uniform lattice then X is a symmetric space of rank one (by [13], Corollary 9.2.2); therefore, we are interested in spaces which do not have uniform lattices, i.e., the universal covering of finite volume, negatively curved manifolds which are not locally symmetric.

(1) Given two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, we say that f is asymptotically equivalent to g when $\lim_{R\to+\infty} f(R)/g(R) = 1$, and we will write $f \sim g$.

(2) We will systematically use the following convenient notation in the paper: given two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, we write $f \overset{C}{\prec} g$ for $R > R_0$ (or $g \overset{C}{\succ} f$) if there exists $C > 0$ such that $f(R) \leq Cg(R)$ for these values of R. Similarly, $f \overset{C}{\asymp} g$ means $g \overset{C}{\prec} f \overset{C}{\prec} g$. We simply write $f \prec g$ and $f \asymp g$ when the constants C and R_0 are unessential.

The *upper* and *lower exponential growth rates* of the function f are respectively defined as:

$$\omega^+(f) = \limsup_{R\to+\infty} R^{-1} \ln f(R) \quad \text{and} \quad \omega^-(f) = \omega(f) = \liminf_{R\to+\infty} R^{-1} \ln f(R)$$

and we simply write $\omega(f)$ when the two limits coincide.

Finally, we say that f is *purely exponential* if $f \asymp e^{\omega(f)R}$, *lower-exponential* when $\limsup_{R\to+\infty} \frac{f(R)}{e^{\omega(f)R}} = 0$ and *upper-exponential* when $\liminf_{R\to+\infty} \frac{f(R)}{e^{\omega(f)R}} = +\infty$; when the weaker property $\limsup_{R\to+\infty} \frac{f(R)}{e^{\omega(f)R}} = +\infty$ holds, we say that f is *weakly upper-exponential*.

It is worth to stress here that the orbital function of Γ is closely related to the volume growth function of X, but it generally has, even for lattices, a different asymptotic behavior than $v_X(x,R)$. A precise asymptotic equivalence for v_Γ was proved by T. Roblin [22] in a very general setting, proving a dichotomy based on the finiteness of the so-called Bowen-Margulis measure associated to Γ. In order to state this dichotomy, let us recall some general definitions. The limit set of a general non elementary discrete subgroup Γ of isometries of X is the subset $\Lambda(\Gamma) \subset X(\infty)$ of accumulation points of any orbit $\Gamma \cdot x$ in X. By Patterson's construction (see [22] for a precise description), there exists on $\Lambda(\Gamma)$ a family $(\mu_x)_{x\in X}$ of finite measures, supported by $\Lambda(\Gamma)$, satisfying the following conditions: for any $x, x' \in X$ and any $g \in \Gamma$,

$$\frac{\mathrm{d}\mu_x}{\mathrm{d}\mu_{x'}}(\xi) = e^{-\delta(\Gamma)b_\xi(x,x')} \quad \text{and} \quad \mu_{\gamma^{-1}x} = \gamma_* \mu_x$$

where $b_\xi(x,x') = \lim_{y\to\xi} d(x,y) - d(x',y)$ is the Busemann function centered at $\xi \in \Lambda(\Gamma)$. When identifying the unit tangent bundle of X with $(X(\infty)\times X(\infty) - D)\times\mathbb{R}$ (where D denotes here the diagonal in $X(\infty)\times X(\infty)$), these two properties readily imply that the measure $e^{-\delta(\Gamma)(b_\eta(x,y)+b_\xi(x,y))}\, d\eta\, d\xi\, dt$ is a Radon measure on UX, which is invariant under the actions of both Γ and the translation flow on the third coordinate; thus, it induces a measure μ_{BM} on the unit tangent bundle $U\bar{X}$ of the quotient manifold $\bar{X} = \Gamma\backslash X$, which is invariant for the geodesic flow and called the *Bowen-Margulis measure*.

T. Roblin proved that for any non-elementary discrete group of isometries Γ of a CAT(-1) space X with non-arithmetic length spectrum [(3)], one has:

(a) $v_\Gamma(x,y,R) \sim c_\Gamma(x,y)e^{\delta(\Gamma)R}$ *if the measure* μ_{BM} *is finite;*

(b) $v_\Gamma(x,y,R) = o(R)e^{\delta(\Gamma)R}$*, where* $o(R)$ *is infinitesimal, otherwise.*

Thus, the behavior of $v_\Gamma(x,R)$ strongly depends on the finiteness of the measure μ_{BM}; also, the asymptotic constant can be expressed in terms of μ_{BM} and of the family of Patterson-Sullivan measures (μ_x) of Γ, as $c_\Gamma(x,y) = \frac{\|\mu_x\|\,\|\mu_y\|}{\delta(\Gamma)\cdot\|\mu_{BM}\|}$.

In this paper we restrict our attention to lattices Γ, which are fundamental examples of *geometrically finite groups*; let us describe this class. Let $C(\Gamma)$ be the convex hull in X of the limit set $\Lambda(\Gamma)$ in $X \cup X(\infty)$; the group Γ acts properly discontinuously on $C(\Gamma)$, the quotient $\bar{N}(\Gamma) = \Gamma\backslash C(\Gamma)$ is called the *Nielsen core* of $\bar{X}$. The group Γ (or the quotient manifold $\bar{X}$) is said to be *geometrically finite* when for some $\varepsilon > 0$ the ε-neighborhood $\bar{N}_\varepsilon(\Gamma)$ of $\bar{N}(\Gamma)$ has finite volume. We refer to [3] for a complete description of geometrical finiteness in variable negative curvature. Some bright examples of non-geometrically finite groups are $\mathbb{Z}^d$-covering of (compact) negatively curved manifolds, whose Bowen-Margulis measure is infinite but for which very refined counting estimates exist [21].

When Γ is a lattice, the Nielsen core $\bar{N}(\Gamma)$ equals $\bar{X}$, thus Γ is clearly geometrically finite. In section §4 we will recall a useful criterion (*Finiteness Criterion* (16), due to Dal'Bo-Otal-Peigné), to decide whether a geometrically finite group has finite Bowen-Margulis measure or not; hence, a precise asymptotics for $v_\Gamma(x,R)$ as in (a).

On the other hand, any convergent group Γ exhibits a behavior as in (b), since it certainly has infinite Bowen-Margulis measure (by Poincaré recurrence, $\mu_{BM}(U\bar{X}) < \infty$ implies that

(3) This means that the additive subgroup of $\mathbb{R}$ generated by the length of closed geodesics in $G\backslash X$ is dense in $\mathbb{R}$; it is the case, for instance, if $dim(X) = 2$, or when $G = \Gamma$ is a lattice.