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CYCLOTOMIC DOUBLE AFFINE HECKE ALGEBRAS

BY Alexander BRAVERMAN, Pavel ETINGOF
and Michael FINKELBERG
with an appendix by Hiraku NAKAJIMA
and Daisuke YAMAKAWA

To Ivan Cherednik with admiration

Abstract. – We show that the partially spherical cyclotomic rational Cherednik algebra (obtained from the full rational Cherednik algebra by averaging out the cyclotomic part of the underlying reflection group) has four other descriptions: (1) as a subalgebra of the degenerate DAHA of type A given by generators; (2) as an algebra given by generators and relations; (3) as an algebra of differential-reflection operators preserving some spaces of functions; (4) as equivariant Borel-Moore homology of a certain variety. Also, we define a new q-deformation of this algebra, which we call *cyclotomic DAHA*. Namely, we give a q-deformation of each of the above four descriptions of the partially spherical rational Cherednik algebra, replacing differential operators with difference operators, degenerate DAHA with DAHA, and homology with K-theory, and show that they give the same algebra. In addition, we show that spherical cyclotomic DAHA are quantizations of certain multiplicative quiver and bow varieties, which may be interpreted as K-theoretic Coulomb branches of a framed quiver gauge theory. Finally, we apply cyclotomic DAHA to prove new flatness results for various kinds of spaces of q-deformed quasiinvariants.

Résumé. – Nous démontrons que l'algèbre rationnelle cyclotomique de Cherednik partiellement sphérique (obtenue à partir de l'algèbre rationnelle de Cherednik complète en effectuant la moyenne par la partie cyclotomique du groupe de réflexions sous-jacent) admet quatre autres descriptions: (1) comme une sous-algèbre de la DAHA degenerée de type A donnée par générateurs; (2) comme une algèbre donnée par générateurs et relations; (3) comme une algèbre des opérateurs différentiels-réflexions préservants certains espaces des fonctions; (4) comme l'homologie de Borel-Moore équivariante d'une certaine variété. Aussi nous définissons une nouvelle q-déformation de cette algèbre que nous appelons *DAHA cyclotomique*. À savoir, nous donnons une q-déformation de chacune des descriptions ci-dessus de l'algèbre rationelle de Cherednik partiellement sphérique, remplaçant les opérateurs différentiels par les opérateurs en différences, DAHA dégenerée par DAHA, et l'homologie par la K-théorie; et démontrons qu'ils donnent lieu à la même algèbre. En outre, nous montrons que les DAHA sphériques cyclotomiques sont les quantifications de certaines variétés de carquois et arc multiplicatives, qui peuvent être interprétées comme les branches de Coulomb K-théoriques d'une théorie de jauge de carquois encadrée. Enfin, nous appliquons la DAHA cyclotomique pour prouver de nouveaux résultats de platitude pour des types différents d'espaces de quasi-invariants q-déformés.

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1. Introduction

Let $N \geq 0, l \geq 0$ be integers, $c_0, \ldots, c_{l-1}, \hbar, k$ be parameters, and $c = (c_0, \ldots, c_{l-1})$. Let $\mathbb{H}_N^{l,\mathrm{cyc}}(c, \hbar, k)$ be the cyclotomic rational Cherednik algebra attached to the complex reflection group $W = S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$. Let $\mathbf{p}$ be the symmetrizer of the subgroup $(\mathbb{Z}/l\mathbb{Z})^N$, and $\mathbb{H}_N^{l,\mathrm{psc}}(c, \hbar, k) := \mathbf{p}\mathbb{H}_N^{l,\mathrm{cyc}}(c, \hbar, k)\mathbf{p}$ be the corresponding partially spherical subalgebra.

In this paper we give a geometric interpretation of $\mathbb{H}_N^{l,\mathrm{psc}}$ as the equivariant Borel-Moore homology of a certain variety $\mathcal{R} = \mathcal{R}(N, l)$ equipped with a group action. This allows us to define a natural q-deformation HH_N^l of $\mathbb{H}_N^{l,\mathrm{psc}}$ in terms of the equivariant K-theory of $\mathcal{R}(N, l)$, which we call the *cyclotomic double affine Hecke algebra* (DAHA).

The existence of this q-deformation may seem somewhat surprising from the viewpoint of classical algebraic theory of DAHA ([15]), since typically DAHA are attached to crystallographic reflection groups (Weyl groups), while the group W is not crystallographic for $l \geq 3$. Yet, we also give a purely algebraic definition of cyclotomic DAHA. Namely, we characterize the cyclotomic DAHA as the subalgebra of the usual Cherednik's DAHA for GL_N generated by certain elements, and also as the subalgebra preserving certain spaces of functions. Finally, we present cyclotomic DAHA by generators and relations. These three descriptions also make sense in the trigonometric limit $q \to 1$ (for partially spherical cyclotomic rational Cherednik algebras). We note that for $l = 1$, cyclotomic DAHA essentially appeared in [3].

We also connect cyclotomic DAHA with multiplicative quiver and bow varieties. Namely, we show that the spherical cyclotomic DAHA $\mathbf{e}HH_N^l(Z, 1, t)\mathbf{e}$ (where $\mathbf{e}$ is the symmetrizer of the finite Hecke algebra) is commutative, and its spectrum for generic parameters is isomorphic to the algebra of regular functions on the multiplicative quiver variety for the cyclic quiver of length l with dimension vector $(N, \ldots, N)$; hence $\mathbf{e}HH_N^l(Z, q, t)\mathbf{e}$ is a quantization of this variety. In particular, we show that this multiplicative quiver variety is connected, and that $HH_N^l(Z, 1, t)$ is an Azumaya algebra of degree $N!$ over this variety. We also show that if t is not a root of unity then the algebra $\mathbf{e}HH_N^l(Z, 1, t)\mathbf{e}$ is an integrally closed Cohen-Macaulay domain isomorphic to the center $\mathcal{Z}(HH_N^l(Z, 1, t))$, while $HH_N^l(Z, 1, t)\mathbf{e}$ is a Cohen-Macaulay module over this algebra.

Finally, we provide some applications of cyclotomic DAHA to the theory of quasiinvariants. Namely, we show that natural q-deformations of various classes of spaces of quasiinvariants are flat, and therefore free modules over the algebra of symmetric polynomials. We also introduce a new type of quasiinvariants (namely, twisted quasiinvariants) and their q-deformation, and prove the freeness property for them.

We note that the degenerate cyclotomic DAHA were studied in a way similar to ours by R. Kodera and H. Nakajima in [30]. In fact, their paper was one of the starting points for our work.

The paper is organized as follows. In Section 2 we develop the theory of partially spherical cyclotomic rational Cherednik algebras as subalgebras in trigonometric (degenerate) DAHA, and give their presentation. In Section 3 we define cyclotomic DAHA as subalgebras of DAHA, and study their properties. We also give a presentation of cyclotomic DAHA, which allows us to find various bases in them and prove flatness results. In Section 4 we give a geometric description of cyclotomic DAHA and their degenerate versions in terms of equivariant K-theory and Borel-Moore homology, and apply it to proving flatness of these algebras. In Section 5 we relate the spherical subalgebra of cyclotomic DAHA at $q = 1$ with certain multiplicative quiver and bow varieties; the latter are isomorphic to the K-theoretic Coulomb branch of framed quiver gauge theories of affine type A. We also study the properties of the spectrum of the spherical cyclotomic DAHA for $q = 1$. In Section 6 we give applications of cyclotomic DAHA to proving flatness of q-deformation of various spaces of quasi-invariants. Finally, Appendix , written by H. Nakajima and D. Yamakawa, explains the relations between multiplicative bow varieties and (various versions of) multiplicative quiver varieties for a cyclic quiver.

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2. Degenerate cyclotomic DAHA

2.1. Notation

In this paper, we will consider many different algebras depending on parameters. So let us clarify our conventions.

First of all, if an algebra depends on parameters, we will list the parameters explicitly when they are given numerical values, and omit them when they are indeterminates (i.e., we work over a commutative base algebra generated by them). Also, throughout the paper, we will use the following notation, to be defined below.

- $HH_{N,\mathrm{deg}}(\hbar, k)$: the degenerate (trigonometric) DAHA, Definition 2.1;
- $HH^{l}_{N,\mathrm{deg}}(z, \hbar, k)$, $z := (z_1, \dots, z_l)$: the degenerate cyclotomic DAHA, Definition 2.8;
- $\mathbb{HH}^{l,\mathrm{cyc}}_{N}(c, \hbar, k)$, $c := (c_0, \dots, c_{l-1})$: the cyclotomic rational Cherednik algebra for the group $S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$, Definition 2.16;
- $\mathbb{HH}^{l,\mathrm{psc}}_{N}(c, \hbar, k) := \mathbf{p}\mathbb{HH}^{l,\mathrm{cyc}}_{N}(c, \hbar, k)\mathbf{p}$: the partially spherical cyclotomic rational Cherednik algebra, Subsection 2.8;
- $HH_N(q, t)$: Cherednik's DAHA, Definition 3.1;
- $HH^{\mathrm{formal}}_{N}(\hbar, k)$: formal Cherednik's DAHA over $\mathbb{C}[[\varepsilon]]$, with $q = e^{\varepsilon\hbar}$ and $t = e^{-\varepsilon k}$, Subsection 3.1;