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Robert J. BERMAN & Tamás DARVAS & Chinh H. LU

*Regularity of weak minimizers of the K-energy and applications to  
properness and K-stability*

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# REGULARITY OF WEAK MINIMIZERS OF THE K-ENERGY AND APPLICATIONS TO PROPERNESS AND K-STABILITY

BY ROBERT J. BERMAN, TAMÁS DARVAS AND CHINH H. LU

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ABSTRACT. – Let  $(X, \omega)$  be a compact Kähler manifold and  $\mathcal{H}$  the space of Kähler metrics cohomologous to  $\omega$ . If a cscK metric exists in  $\mathcal{H}$ , we show that all finite energy minimizers of the extended K-energy are smooth cscK metrics, partially confirming a conjecture of Y.A. Rubinstein and the second author. As an immediate application, we obtain that the existence of a cscK metric in  $\mathcal{H}$  implies J-properness of the K-energy, thus confirming one direction of a conjecture of Tian. Exploiting this properness result we prove that an ample line bundle  $(X, L)$  admitting a cscK metric in  $c_1(L)$  is K-polystable. When the automorphism group is finite, the properness result, combined with a result of Boucksom-Hisamoto-Jonsson, also implies that  $(X, L)$  is uniformly K-stable.

RÉSUMÉ. – Soient  $(X, \omega)$  une variété kählérienne compacte et  $\mathcal{H}$  l'espace des métriques de Kähler dans la classe de cohomologie de  $\omega$ . S'il existe une métrique cscK dans  $\mathcal{H}$ , nous montrons que tous les minimiseurs dans l'espace d'énergie finie de la fonctionnelle K-énergie sont lisses, confirmant partiellement une conjecture de Y.A. Rubinstein et le deuxième auteur. Comme une conséquence immédiate, nous en déduisons que l'existence d'une métrique cscK dans  $\mathcal{H}$  implique la J-propreté de la fonctionnelle K-énergie. Ceci confirme une direction de la conjecture de Tian. En utilisant cette propriété nous montrons qu'un fibré ample  $(X, L)$  admettant une métrique cscK dans  $c_1(L)$  est K-polystable. Quand le groupe d'automorphisme est fini le résultat de propriété, combiné avec un résultat récent de Boucksom-Hisamoto-Jonsson, implique aussi que  $(X, L)$  est uniformément K-stable.

## 1. Introduction and main results

Let  $(X, J, \omega)$  be a compact connected Kähler manifold. By

$$\mathcal{H}_\omega = \{v \in \mathcal{C}^\infty(X) \mid \omega_v := \omega + i\partial\bar{\partial}v > 0\}$$

we denote the space of Kähler potentials. By the  $\partial\bar{\partial}$ -lemma of Hodge theory, up to a constant, this space is in a one-to-one correspondence with  $\mathcal{H}$ , the space of Kähler metrics cohomologous to  $\omega$ . The problem of finding canonical metrics in  $\mathcal{H}$  goes back to Calabi in the fifties. In this work we will point necessary conditions under which  $\mathcal{H}$  admits constant scalar curvature Kähler (cscK) metrics, in terms of energy properness.

We now elaborate on the terminology necessary to state our main results. To have a one-to-one correspondence between potentials and metrics, we consider the space

$$\mathcal{H}_0 := \mathcal{H}_\omega \cap \text{AM}^{-1}(0),$$

and we always work on the level of potentials unless specified otherwise (for the definition of AM see (3) below). The connected Lie group of holomorphic automorphisms

$$G := \text{Aut}_0(X, J)$$

acts naturally on  $\mathcal{H}$  via pullbacks, hence it also acts on  $\mathcal{H}_0$  (see [22, Section 5.2] for a precise description of this action on the level of potentials).

Motivated by results and ideas in conformal geometry, in the 90's Tian introduced the notion of “J-properness” on  $\mathcal{H}_\omega$  [45, Definition 5.1] in terms of Aubin's nonlinear energy functional  $J_\omega$  and the Mabuchi K-energy  $E$ . This condition says that for any  $u_j \in \mathcal{H}_\omega$  we have

$$(1) \quad J_\omega(u_j) \rightarrow \infty \quad \text{implies} \quad E(u_j) \rightarrow \infty.$$

We refer to Section 2 for the precise definitions of  $J_\omega$  and  $E$ .

Tian conjectured that existence of constant scalar curvature Kähler (csck) metrics in  $\mathcal{H}_\omega$  should be equivalent to J-properness of the K-energy  $E$  [45, Remark 5.2],[47] and this was proved for Fano manifolds with  $G$  trivial [46, 50]. In [36, Theorem 1] the “strong form” of the J-properness condition (1) was obtained, confirming another conjecture of Tian from [46] (for Fano manifold with trivial  $G$ ), saying that the K-energy grows at least linearly with respect to the J-functional. This stronger form has been later adopted in the literature, sometimes referred to as “coercivity”.

When  $G$  is non-trivial it was known that the conjecture cannot, in general, hold as stated above and numerous modifications were proposed by Tian (see [47, Conjecture 7.12], [48]). In [22], Y.A. Rubinstein and the second named author disproved one of these conjectures, proved the remaining ones for general Fano manifolds, and the following conjecture was stated for general Kähler manifolds:

**CONJECTURE 1.1** (Conjecture 2.8 in [22]). – *Suppose  $(X, \omega)$  is a Kähler manifold. There exists a csck metric cohomologous to  $\omega$  if and only if for some  $C, D > 0$  we have*

$$E(u) \geq C \inf_{g \in G} J_\omega(g.u) - D, \quad u \in \mathcal{H}_\omega.$$

This “modified properness conjecture” thus reduces to Tian's original prediction in case  $G$  is trivial and was originally stated for Fano manifolds by Tian himself [48]. It was proved in this context (of Fano manifolds) in [22, Theorem 2.4], and this paper also linked the resolution of the general conjecture to a regularity question on weak minimizers of the K-energy that we elaborate now.

We denote by  $(\mathcal{E}^1, d_1)$  the metric completion of  $\mathcal{H}_\omega$  with respect to the  $L^1$ -type Mabuchi path length metric  $d_1$ . We refer to Sections 2.1-2.2 for more precise details about this metric structure introduced in [19]. The point of connection with the questions investigated here is the fact that  $d_1$  metric growth is comparable to  $J_\omega$  [22, Proposition 5.5], and we refer to [22, Section 4, Section 5] for a more detailed exposition on how the  $d_1$ -metric geometry relates

to J-properness. Let us now state the regularity conjecture of [22] (see [22, Conjecture 2.9]) and the theorem that connects it to Conjecture 1.1 above:

CONJECTURE 1.2 (Conjecture 2.9 in [22]). – *Suppose  $(X, \omega)$  is a compact Kähler manifold. The minimizers of the extended K-energy  $E : \mathcal{E}^1 \rightarrow (-\infty, +\infty]$  are smooth cscK metrics.*

THEOREM 1.3 (Theorem 2.10 in [22]). – *Conjecture 1.2 implies Conjecture 1.1.*

Our first main result partially confirms Conjecture 1.2 and also a less general conjecture of X.X. Chen [17, Conjecture 6.3]:

THEOREM 1.4. – *Suppose  $(X, \omega)$  is a cscK manifold. If  $v \in \mathcal{E}^1$  minimizes the extended K-energy  $E : \mathcal{E}^1 \rightarrow (-\infty, +\infty]$ , then  $v$  is a smooth cscK potential. In particular there exists  $g \in G$  such that  $g^*\omega_v = \omega$ .*

The last claim follows from the uniqueness result of [8]. Using this result and Theorem 1.3 we immediately obtain one direction of Conjecture 1.1:

THEOREM 1.5. – *Suppose  $(X, \omega)$  is a cscK manifold. Then for some  $C, D > 0$  we have*

$$(2) \quad E(u) \geq C \inf_{g \in G} J_\omega(g.u) - D, \quad u \in \mathcal{H}_\omega.$$

The proof of Theorem 1.4 relies on the  $L^1$ -Mabuchi geometry of  $\mathcal{H}_\omega$  introduced in [20, 19], the finite energy pluripotential theory of [9, 28] and the convexity methods of [22, 12] and [8]. Realizing that the metric geometry of  $\mathcal{H}_\omega$  and J-properness should be related seems to have first appeared in [17, Conjecture 6.1], but this work rather proposed the use of the  $L^2$ -Mabuchi metric on  $\mathcal{H}_\omega$ .

As a consequence of Theorem 1.5 and the techniques of [46, 6] we obtain a result on K-polystability, originally proved by Mabuchi ([33, Main Theorem] see also [32]), using a completely different argument. Slightly less general, or different flavor results were obtained by Stoppa, Stoppa-Székelyhidi, Székelyhidi [40, Theorem 1.2], [41, Theorem 1.4], [43, Theorem A] and others. We recall the relevant terminology in the last section of the paper.

THEOREM 1.6. – *Suppose  $L \rightarrow X$  is a positive line bundle. If there exists a cscK metric in the class  $c_1(L)$ , then  $(X, L)$  is K-polystable.*

The idea of proving K-stability via properness goes back to Tian's seminal paper [46]. The main point of our approach, involving geodesic rays, is to generalize the findings of [6] from the Fano case.

In case the group  $G$  is trivial, the results in [11, 13, 24] show that properness implies uniform K-stability in the  $L^1$ -sense (for terminology, see [11, 13, 24] and references therein). Thus, as a consequence of Theorem 1.5 we obtain the following:

COROLLARY 1.7. – *Assume that  $(X, L)$  is a positive line bundle and  $G$  is trivial. If there exists a cscK metric in  $c_1(L)$ , then  $(X, L)$  is uniformly K-stable.*