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COHOMOLOGICAL RANK FUNCTIONS ON ABELIAN VARIETIES

BY Zhi JIANG AND Giuseppe PARESCHI

ABSTRACT. – Generalizing the *continuous rank function* of Barja-Pardini-Stoppino, in this paper we consider *cohomological rank functions* of $\mathbb{Q}$-twisted (complexes of) coherent sheaves on abelian varieties. They satisfy a natural transformation formula with respect to the Fourier-Mukai-Poincaré transform, which has several consequences. In many concrete geometric contexts these functions provide useful invariants. We illustrate this with two different applications, the first one to GV-subschemes and the second one to multiplication maps of global sections of ample line bundles on abelian varieties.

RÉSUMÉ. – En généralisant la fonction rang continu de Barja-Pardini-Stoppino, nous considérons dans cet article les fonctions rang cohomologiques des (complexes de) faisceaux cohérents sur les variétés abéliennes. Ils répondent à une formule de transformation naturelle par rapport à la transformée de Fourier-Mukai-Poincaré, ce qui a plusieurs conséquences. Dans de nombreux contextes géométriques concrets, ces fonctions fournissent des invariants utiles. Nous illustrons ceci avec deux applications différentes, la première pour les sous-schémas GV et la seconde pour la multiplication de sections globales des fibrés en droites amples sur les variétés abéliennes.

Introduction

In their paper [3] M.A. Barja, R. Pardini and L. Stoppino introduce and study the *continuous rank function* associated to a line bundle M on a variety X equipped with a morphism $X \xrightarrow{f} A$ to a polarized abelian variety. Motivated by their work, we consider more generally *cohomological rank functions*—defined in a similar way—of a bounded complex $\mathcal{F}$ of coherent sheaves on a polarized abelian variety $(A, \underline{l})$ defined over an algebraically closed field of characteristic zero. As it turns out, these functions often encode interesting geometric

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information. The purpose of this paper is to establish some general structure results about them and show some examples of application.

Let L be an ample line bundle on an abelian variety A, let $\underline{l} = c_1(L)$ and let $\varphi_{\underline{l}} : A \to \widehat{A}$ be the corresponding isogeny. The cohomological rank functions of $\mathscr{F} \in \mathrm{D}^b(A)$ with respect to the polarization $\underline{l}$ are initially defined (see Definition 2.1 below) as certain continuous rational-valued functions

$$h^i_{\mathscr{F},\underline{l}} : \mathbb{Q} \to \mathbb{Q}^{\geq 0}.\ ^{(1)}$$

The definition of these functions is peculiar to abelian varieties (and more generally to irregular varieties), as it uses the isogenies $\mu_b : A \to A$, $z \mapsto bz$. For $x \in \mathbb{Z}$, $h^i_{\mathscr{F},\underline{l}}(x) := h^i_{\mathscr{F}}(x\underline{l})$ coincides with the generic value of $h^i(A, \mathscr{F} \otimes L^x)$, for L varying among all line bundles representing $\underline{l}$. This is extended to all $x \in \mathbb{Q}$ using the isogenies μ_b. In fact the rational numbers $h^i_{\mathscr{F}}(x\underline{l})$ can be interpreted as generic cohomology ranks of the $\mathbb{Q}$-twisted coherent sheaf (or, more generally, $\mathbb{Q}$-twisted complex of coherent sheaves) $\mathscr{F}\langle x\underline{l}\rangle$ (in the sense of Lazarsfeld [15, §6.2A]).

The above functions are closely related to the Fourier-Mukai transform $\Phi_{\mathscr{P}} : \mathrm{D}^b(A) \to \mathrm{D}^b(\widehat{A})$ associated to the Poincaré line bundle and our first point consists in exploiting systematically such a relation. We prove the following transformation formula (Proposition 2.3 below):

$$h^i_{\mathscr{F}}(x\underline{l}) = \frac{(-x)^g}{\chi(\underline{l})} h^i_{\varphi_{\underline{l}}^* \Phi_{\mathscr{P}}(\mathscr{F})}(-\frac{1}{x}\underline{l}) \qquad \text{for } x \in \mathbb{Q}^-, \tag{0.1}$$

$$h^i_{\mathscr{F}}(x\underline{l}) = \frac{x^g}{\chi(\underline{l})} h^{g-i}_{\varphi_{\underline{l}}^* \Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)}(\frac{1}{x}\underline{l}) \qquad \text{for } x \in \mathbb{Q}^+. \tag{0.2}$$

This has several consequences, summarized in the following theorem. The proof and discussion of the various items are found in Sections 2,3 and 4.

Theorem A. – *Let* $\mathscr{F} \in \mathrm{D}^b(A)$ *and* $i \in \mathbb{Z}$. *Let* $g = \dim A$.
(1) (Corollaries 2.4, 2.6, 2.7.) *For each* $x_0 \in \mathbb{Q}$ *there are* $\epsilon^-, \epsilon^+ > 0$ *and two* (explicit, see below) *polynomials* $P^+_{i,\mathscr{F},x_0}, P^-_{i,\mathscr{F},x_0} \in \mathbb{Q}[x]$ *of degree* $\leq g$ *such that* $P^+_{i,\mathscr{F},x_0}(x_0) = P^-_{i,\mathscr{F},x_0}(x_0)$ *and*

$$h^i_{\mathscr{F}}(x\underline{l}) = P^-_{i,\mathscr{F},x_0}(x) \quad \textit{for } x \in (x_0 - \epsilon^-, x_0] \cap \mathbb{Q},$$
$$h^i_{\mathscr{F}}(x\underline{l}) = P^+_{i,\mathscr{F},x_0}(x) \quad \textit{for } x \in [x_0, x_0 + \epsilon^+) \cap \mathbb{Q}.$$

(2) (Proposition 4.4) *Let* $k < g$ *and* $x_0 \in \mathbb{Q}$. *If the function* $h^i_{\mathscr{F},\underline{l}}$ *is strictly of class* $\mathcal{C}^k$ *at* x_0 *then the jump locus* $J^{i+}(\mathscr{F}\langle x_0\rangle)$ (see §4 for the definition) *has codimension* $\leq k+1$.
(3) (Theorem 3.2) *The function* $h^i_{\mathscr{F},\underline{l}}$ *extends to a continuous function* $h^i_{\mathscr{F},\underline{l}} : \mathbb{R} \to \mathbb{R}^{\geq 0}$. (2)

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(1) In the present context the above-mentioned continuous rank function of Barja-Pardini-Stoppino is recovered as $h^0_{f_*M,\underline{l}}$ (see the notation above).

(2) This theorem provides partial answers to some questions raised, in the specific case of the above mentioned continuous rank functions $h^0_{f_*M,\underline{l}}$, in [3], e.g., Question 8.11. We also point out that for such functions item (3) of the present theorem, as well as some additional properties, were already proved in loc. cit. via different methods.

It follows from (1) that for $x_0 \in \mathbb{Q}$ the function $h^i_{\mathcal{F},\underline{l}}$ is smooth at x_0 if and only if the two polynomials $P^-_{i,\mathcal{F},x_0}$ and $P^+_{i,\mathcal{F},x_0}$ coincide. If this is not the case x_0 is called a *critical point*.

It turns out (Corollary 2.4) that for $x_0 \in \mathbb{Z}$ the two polynomials $P^-_{i,\mathcal{F}\,x_0}(x)$ and $P^+_{i,\mathcal{F},x_0}(x)$ are obtained from the Hilbert polynomials (with respect to the polarization $\underline{l}$) of the two coherent sheaves $\mathcal{G}^{i,+}_{x_0} := \varphi^*_{\underline{l}} R^i\Phi_{\mathcal{P}}(\mathcal{F} \otimes L^{x_0})$ and $\mathcal{G}^{i,-}_{x_0} := \varphi^*_{\underline{l}} R^{g-i}\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee \otimes L^{-x_0})$ in the following way:

$$P^-_{i,x_0,\mathcal{F}}(x) = \frac{(-x)^g}{\chi(\underline{l})}\chi_{\mathcal{G}^{i,+}_{x_0}}(-\frac{1}{x}\underline{l}), \tag{0.3}$$

$$P^+_{i,x_0,\mathcal{F}}(x) = \frac{x^g}{\chi(\underline{l})}\chi_{\mathcal{G}^{i,-}_{x_0}}(\frac{1}{x}\underline{l}). \tag{0.4}$$

For non-integer $x_0 \in \mathbb{Q}$ the two polynomials $P^-_{i,\mathcal{F}\,x_0}(x)$ and $P^+_{i,\mathcal{F},x_0}(x)$ have a similar description after reducing to the integer case (Corollary 2.6). Thus item (2) of Theorem A tells that, for $x_0 \in \mathbb{Q}$, the first k coefficients of the polynomials $P^-_{i,x_0,\mathcal{F}}(x)$ and $P^+_{i,x_0,\mathcal{F}}(x)$ coincide as soon as the rank function $\mathrm{Pic}^0 A \to \mathbb{Z}^{\geq 0}$ defined by $\alpha \mapsto h^i(\mathcal{F} \otimes L^{x_0} \otimes P_\alpha)$ has jump locus of codimension $\geq k+1$. In this last formulation we are implicitly assuming that x_0 is integer but for rational x_0 the situation is completely similar. However there might be irrational critical points (see e.g., Example 4.1), and at present we lack any similar interpretation for them.

In Section 5 we relate cohomological rank functions with the notions of GV, M-regular and IT(0)-sheaves, which are extended here to the $\mathbb{Q}$-twisted setting. We provide formulations of Hacon's results ([9]), and some related ones, which are simpler and more convenient even for usual sheaves. Finally, in Section 6 we point out some integral properties of cohomological rank functions.

It seems that the critical points of the function and the polynomials $P^-_{i,\mathcal{F},x_0}$ and $P^+_{i,\mathcal{F},x_0}$ are interesting and sometimes novel invariants in many concrete geometric situations. We exemplify this in the following two applications.

Application to GV-subschemes. – Our first example concerns GV-subschemes of principally polarized abelian varieties (here we will assume that the ground field is $\mathbb{C}$). This notion (we refer to Section 7 below for the definition and basic properties) was introduced in [24] in the attempt of providing a Fourier-Mukai approach to the minimal class conjecture ([6]), predicting that the only effective algebraic cycles representing the minimal classes $\frac{\theta^{g-d}}{(g-d)!} \in H^{2(g-d)}(A,\mathbb{Z})$ are (translates of) the subvarieties $\pm W_d(C)$ of Jacobians $J(C)$, and $\pm F$, the Fano surface in the intermediate Jacobian of a cubic threefold.

It is known that the subvarieties $W_d(C)$ of Jacobians, as well as the Fano surface ([10]) are GV-subschemes and that, on the other hand, geometrically non-degenerate GV-subschemes have minimal classes ([24]). Therefore it was conjectured in loc. cit. that geometrically non-degenerate GV-subschemes are either (translates of) $\pm W_d(C)$ or $\pm F$ as above. Denoting g the dimension of the p.p.a.v. and d the dimension of the subscheme, this is known only in a few cases: (i) for $d = 1$ and $d = g-2$ (loc. cit.); (ii) for $g = 5$, settled in the recent work [5], (iii) for Jacobians and intermediate Jacobians of generic cubic threefolds, as consequences of the main results of respectively [6] and [11]. In the recent work [28] it is