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PERIODIC TRIVIAL EXTENSION ALGEBRAS AND FRACTIONALLY CALABI-YAU ALGEBRAS

BY Aaron CHAN, Erik DARPÖ, Osamu IYAMA AND René MARCZINZIK

Dedicated to the memory of Andrzej Skowroński

ABSTRACT. – We study periodicity and twisted periodicity of the trivial extension algebra $T(A)$ of a finite-dimensional algebra A. We show that (twisted) periodicity of $T(A)$ is equivalent to A being (twisted) fractionally Calabi-Yau of finite global dimension. We also extend this result to a large class of self-injective orbit algebras. As a consequence, the results give a partial answer to the periodicity conjecture of Erdmann-Skowroński, which expects the classes of periodic and twisted periodic algebras to coincide. On the practical side, it allows us to construct a large number of new examples of periodic algebras and fractionally Calabi-Yau algebras. We also establish a connection between periodicity and cluster tilting theory, by showing that twisted periodicity of $T(A)$ is equivalent to the d-representation-finiteness of the r-fold trivial extension algebra $T_r(A)$ for some $r, d \geq 1$. This answers a question by Darpö and Iyama.

As applications of our results, we give answers to some other open questions. We construct periodic symmetric algebras of wild representation type with arbitrary large minimal period, answering a question by Skowroński. We also show that the class of twisted fractionally Calabi-Yau algebras is closed under derived equivalence, answering a question by Herschend and Iyama.

RÉSUMÉ. – Nous étudions la périodicité et la périodicité tordue de l'extension triviale $T(A)$ d'une algèbre de dimension finie. Nos montrons que l'extension triviale d'une algèbre est périodique (tordue) si et seulement si cette algèbre est Calabi-Yau fractionnaire (tordue) de dimension globale finie. Nous étendons également ce résultat à une large classe d'algèbres d'orbites auto-injectives. Nous en déduisons une réponse partielle à la conjecture de périodicité d'Erdmann-Skowroński qui prédit que les classes d'algèbres périodiques et périodiques tordues coincident. De manière concrète, cela permet de construire de nombreux nouveaux exemples d'algèbres périodiques et d'algèbres Calabi-Yau fractionnaires. Nous établissons également un lien entre périodicité et théorie de l'amas-basculement en montrant que la périodicité tordue de l'extension triviale $T(A)$ d'une algèbre A est équivalente à l'existence d'entiers r et d pour lesquels l'extension triviale r-pliée $T_r(A)$ est de type d-représentation finie, répondant ainsi à une question de Darpö et Iyama.

Nos résultats permettent également de répondre à d'autres questions ouvertes : nous construisons des algèbres périodiques symétriques de type de représentation sauvage de période minimale arbitrairement grande, ce qui répond à une question de Skowroński et nous montrons que la classe des algèbres

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Calabi-Yau fractionnaires tordues est fermée sous équivalence dérivée, ce qui répond à une question de Herschend et Iyama.

1. Introduction

The syzygy functor $\Omega_A : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ [8] is a fundamental tool in the homological algebra of a Noetherian ring A. A finite-dimensional k-algebra A is said to be *periodic* (of period n) if $\Omega^n_{A^e}(A) \simeq A$ as A^e-modules for some $n \geq 1$, where $A^e = A \otimes_k A^{op}$ is the enveloping algebra of A. This implies the periodicity of the syzygy functor Ω^n_A, and is often a more workable condition than the latter. All periodic algebras are self-injective, and amongst self-injective algebras, the periodic ones constitute a fundamental subclass, with many important properties. The following problem, raised in [26, Problem 1], is central to understanding self-injective algebras, and also highly significant in the theory of Hochschild cohomology and support varieties [42, 25].

Problem 1.1. – *For a self-injective algebra B, when is B periodic?*

For example, preprojective algebras of Dynkin type are periodic. Also, the trivial extension algebra of the path algebra kQ of an acyclic quiver Q is periodic if and only if Q is Dynkin [11]. Periodic algebras appear also in group representation theory, topology and algebraic geometry, e.g., some contraction algebras are periodic [19]. We refer to [26] for a survey on periodic algebras, and to e.g., [5, 10, 21, 22, 27, 28, 54] for more recent contributions.

There are some natural variations of periodicity, see Figure 1 in Section 4. In particular, we call a finite-dimensional k-algebra A *twisted periodic* if $\Omega^n_{A^e}(A) \simeq {}_1A_\phi$ as A^e-modules, for some $n \geq 1$ and k-algebra automorphism ϕ of A (here, ${}_1A_\phi$ denotes the A^e-module A with right action twisted by ϕ). In the recent article [27] the following important question is formulated as a conjecture, and given an affirmative answer for group algebras.

Question 1.2 (Periodicity conjecture [27]). – *Is every finite-dimensional twisted periodic algebra periodic?*

One of the most fundamental classes of self-injective algebras is given by the *trivial extension algebra* $T(A)$ of a finite-dimensional k-algebra A. Recall that $T(A) = A \oplus \mathrm{Hom}_k(A, k)$ with multiplication given by $(a, f)(b, g) = (ab, ag + fb)$. Trivial extension algebras, together with their dg (=differential graded) analogues, have played important roles in the representation theory of algebras. Examples include the classification of (d-)representation-finite and tame self-injective (dg) algebras [85, 47, 77, 79, 81, 16, 57], the study of derived categories [44] and cluster categories [58, 4], gentle algebras and Brauer graph algebras [7, 78]. They also appear in other fields, such as symplectic and contact geometry [61, 32].

The purpose of this paper is to study periodicity and twisted periodicity of the trivial extension algebra $T(A)$, including Problem 1.1 and Question 1.2, and relate it to homological properties of the algebra A. We will give a complete solution to Problem 1.1 in this vein for $B = T(A)$. As an application, we give a large number of new examples of periodic algebras, including many of wild representation type. In Section 7, we extend our solution to a large class of self-injective algebras given by the repetitive category [79, 81, 83, 26].

To explain our solution to Problem 1.1, recall that the bounded derived category $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ of a finite-dimensional algebra A of finite global dimension has a Serre functor ν, see (2.2) below. Such an algebra A is said to be *fractionally Calabi-Yau* (also $\frac{m}{\ell}$*-Calabi-Yau*) if there exist integers $\ell > 0$ and m such that ν^{ℓ} and $[m]$ are isomorphic as functors on $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ [88, §18.6]. For example, the path algebra of a Dynkin quiver is $\frac{h-2}{h}$-Calabi-Yau, where h is the Coxeter number [72]—see Theorem A.1 for a more precise statement. There is also a weaker notion of *twisted* fractionally Calabi-Yau, in which the defining isomorphism of functors is taken up to a twist by an algebra automorphism. There are many important examples of (twisted) fractionally Calabi-Yau algebras, in representation theory [41, 45, 67, 77, 89] as well as in algebraic geometry [37, 62, 33, 64, 46]. They play important roles in various areas, e.g., integrable systems [60, 49], Hochschild cohomology [74] and mathematical physics [13].

Our first main result gives a solution to Problem 1.1 for trivial extension algebras.

Theorem 1.3 (Corollaries 6.2 and 7.3). – *Let A be a finite-dimensional algebra over a field k such that $A/\operatorname{rad} A$ is a separable k-algebra (e.g., when k is perfect). Then the following conditions are equivalent.*

(i) *$T(A)$ is periodic.*

(ii) *A has finite global dimension and is fractionally Calabi-Yau.*

Moreover, let G be an admissible group of automorphisms of the repetitive category $\widehat{A}$ (see Section 2.2) containing $\nu_{\widehat{A}}^{\ell}$ for some $\ell \geq 1$. Then the following condition is equivalent to (i) *and* (ii).

(iii) *The orbit algebra $\widehat{A}/G$ is periodic.*

This gives a large number of new periodic algebras, see Section 8. As a consequence, we get a conceptual proof of the periodicity of the trivial extension algebras of the path algebras of Dynkin quivers mentioned above, see Example 8.2. The trickiest part of the proof of Theorem 1.3 is the implication (ii)⇒(i), which will be shown in Section 6 by using the relative bar resolution of a certain differential graded algebra quasi-isomorphic to $T(A)$.

The classification of periodic symmetric algebras of tame representation type is a well studied subject, e.g., [28, 29, 30]. According to an Oberwolfach talk in January 2020 by Skowroński [80], there is no known example of a family of wild symmetric algebras with unbounded minimal periods. As an application of our results, we can construct many such examples. For example, the trivial extension $T(A)$ of the incidence algebra A of the Boolean lattice with 2^n elements has minimal period $3+n$ if n is odd or the characteristic of k is two, and $2(n+3)$ otherwise (Corollary 8.11). Here $T(A)$ is indeed wild for $n \geq 4$.

In our second main result, we give several characterisations of *twisted* periodicity for $T(A)$. Recall that, for a positive integer d, a finite-dimensional algebra A is said to be *d-representation-finite* if there exists a d-cluster-tilting A-module. For algebras of finite global dimension, this is closely related to the notion of twisted fractionally Calabi-Yau [45] and, for self-injective algebras, to periodicity [25]. Using results from [16], we characterize