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WANDERING FATOU COMPONENTS AND ALGEBRAIC JULIA SETS

BY EUGENIO TRUCCO

ABSTRACT. — We study the dynamics of polynomials with coefficients in a non-Archimedean field K, where K is a field containing a dense subset of algebraic elements over a discrete valued field k. We prove that every wandering Fatou component is contained in the basin of a periodic orbit. We obtain a complete description of the new Julia set points that appear when passing from K to the Berkovich affine line over K. We give a dynamical characterization of polynomials having algebraic Julia sets. More precisely, we establish that a polynomial with algebraic coefficients has algebraic Julia set if every critical element is nonrecurrent.

RÉSUMÉ (*Composantes de Fatou errantes et ensembles de Julia algébriques*)

Nous étudions la dynamique des polynômes à coefficients dans un corps K non-archimédien, où K contient un sous-ensemble dense d'éléments algébriques sur un corps k à valeurs discrètes. Nous montrons que toute composante de Fatou errante est contenue dans le bassin d'une orbite périodique. Nous obtenons une description complète des nouveaux points d'ensemble de Julia qui apparaissent quand on passe de K à la ligne de Berkovich affine sur K. Nous donnons une caractérisation dynamique des polynômes ayant des ensembles de Julia algébriques. Plus précisément, nous établissons qu'un polynôme à coefficients algébriques a un ensemble de Julia algébrique si tout élément critique est non-recurrent.

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1. Introduction

In this paper we study the dynamics of polynomials $P\colon K \to K$ where K is an algebraically closed field of characteristic 0 which is complete with respect to a non-Archimedean absolute value. Moreover, we will assume that there exists a discrete valued field $k \subseteq K$ such that

$$k^{\mathrm{a}} = \{z \in K \mid [k(z) : k] < +\infty\}$$

form a dense subset of K. Examples of such fields are the field $\mathbb{C}_p$ of p-adic numbers and the field, which we will denote by $\mathbb{L}$, which is the completion of an algebraic closure of the field of formal Laurent series with coefficients in $\mathbb{C}$. Dynamics over $\mathbb{C}_p$ naturally arises in number theory and dynamics over $\mathbb{L}$ naturally appears in the study of parameter spaces of complex rational maps [17].

For complex rational maps acting on the Riemann sphere, Sullivan [25] proved, with the aid of quasi-conformal techniques, that every connected component of the Fatou set of a rational map $R \in \mathbb{C}(z)$ of degree ≥ 2 is eventually periodic (Sullivan's No Wandering Domains Theorem). This is no longer true for general non-Archimedean fields. In fact, Benedetto [4] established the existence of p-adic polynomials having wandering (analytic) domains which are not attracted to a periodic orbit. This result heavily relies on the fact that over p-adic fields, whose residual characteristic is $p > 0$, there exists a phenomenon called *wild ramification.*

The aim of this paper is to study the interplay between algebraic and dynamical properties of points in the Julia set of a polynomial. As a consequence, we establish that for *tame* polynomials (see Definition 2.10), that is, for polynomials such that wild ramification does not occur, the dynamics is free of nontrivial wandering domains (see Corollary B below).

Recent developments on the theory of iteration of rational maps over non-Archimedean fields put in evidence that the correct space to study the action of rational maps is the *Berkovich space* (e.g., [1, 2, 10, 13, 19, 20, 21]). The action of a polynomial $P \in K[z]$ extends to the *Berkovich affine line* $\mathbb{A}^{1,\mathrm{an}}_K$ *associated to* K. Moreover, the notions of Julia set (chaotic dynamics) and Fatou set (regular dynamics) also extend to $\mathbb{A}^{1,\mathrm{an}}_K$. Our first main result is a complete description of the new Julia set points that appear when passing from K to $\mathbb{A}^{1,\mathrm{an}}_K$. We will denote by $\mathcal{J}_P$ the Julia set of P. A polynomial is *simple* if its Julia set is a singleton.

THEOREM A. — *Let $P \in K[z]$ be a nonsimple and tame polynomial of degree $d \geq 2$. Then $\mathcal{J}_P \setminus K$ is empty or, there exist finitely many repelling periodic orbits $\mathcal{O}_1, \dots, \mathcal{O}_m \subseteq \mathbb{A}^{1,\mathrm{an}}_K \setminus K$ such that*

$$\mathcal{J}_P \setminus K = \mathrm{GO}(\mathcal{O}_1) \sqcup \cdots \sqcup \mathrm{GO}(\mathcal{O}_m),$$

where $\mathrm{GO}(\mathcal{O}_j)$ *denotes the grand orbit of* $\mathcal{O}_j$ *and* $1 \leq m \leq d-2$.

The previous theorem is first proven for polynomials in $K[z]$ with algebraic coefficients over the field k. Here, we rely on our study of the interplay between the geometry of the Julia set and the underlying algebraic structure (Section 6). For a general tame polynomial with coefficients in K, we use a perturbation technique furnished by a key proposition (Proposition 7.1) inspired by complex polynomial dynamics (e.g., [18]).

Standard techniques (see Proposition 2.16) allow us to deduce the above mentioned nonwandering result from Theorem A. We say that x is in the basin of the periodic orbit $\mathcal{O}$ if $\mathcal{O}$ is the set of limits points of $\{P^n(x) \mid n \in \mathbb{N}\}$.

Corollary B. — *Let* $P \in K[z]$ *be a tame polynomial of degree* ≥ 2. *Then, every wandering Fatou component is in the basin of a periodic orbit.*

Benedetto [3] proved a similar result to Corollary B for rational maps with *algebraic coefficients* over the field of p-adic numbers $\mathbb{Q}_p$ with some slightly different hypothesis.

In terms of k, K and in our language, Theorem B in [5] says that every wandering Fatou component of a rational map with algebraic coefficients over k is in the basin of a periodic orbit. Benedetto asks (question (2) at the end of the introduction of [5]) if this is true for rational maps with coefficients in K, assuming that the characteristic of the residual field of K is zero. Corollary B above gives an affirmative answer to the question posed by Benedetto in the case of polynomials.

It is not known if for any polynomial $P \in \mathbb{Q}_p[z]$ such that $\mathcal{J}_P \cap \mathbb{Q}_p \neq \varnothing$, there exists a classical repelling periodic point of P in $\mathcal{J}_P \cap \mathbb{Q}_p$. In the case of tame polynomials we have the following result.

Corollary C. — *Let* $P \in K[z]$ *be a nonsimple and tame polynomial of degree* $d \geq 2$. *Then the classical Julia set of* P *contains a repelling periodic point.*

In the case of p-adic polynomials, Bézivin [7, Proposition A] proved that if there exists a repelling periodic point in $\mathcal{J}_P \cap \mathbb{Q}_p$ and $\mathcal{J}_P \cap \mathbb{Q}_p$ is a compact set, then $\mathcal{J}_P = \mathcal{J}_P \cap \mathbb{Q}_p$. The following corollary is an analog of that result, nevertheless, we do not need to assume the existence of a repelling periodic point.

Corollary D. — *Let* $P \in K[z]$ *be a tame polynomial of degree* ≥ 2. *Then the following statements are equivalent:*

1. $\mathcal{J}_P \cap K$ *is a compact subset of* $\mathbb{A}_K^{1,\mathrm{an}}$.
2. *There is no critical periodic element in* $\mathcal{J}_P$.
3. $\mathcal{J}_P = \mathcal{J}_P \cap K$.