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**GLOBAL NONLINEAR
STABILITY
OF MINKOWSKI SPACE
FOR SPACELIKE-CHARACTERISTIC
INITIAL DATA**

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**GLOBAL NONLINEAR STABILITY
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GLOBAL NONLINEAR STABILITY OF MINKOWSKI SPACE FOR SPACELIKE-CHARACTERISTIC INITIAL DATA

Olivier Graf

Abstract. – In this paper, we prove the global nonlinear stability of Minkowski space in the context of the spacelike-characteristic Cauchy problem for Einstein vacuum equations. Spacelike-characteristic initial data are posed on a compact 3-disk and on the future complete null hypersurface emanating from its boundary. Our result extends the seminal stability result for Minkowski space proved by Christodoulou and Klainerman for which initial data are prescribed on a spacelike 3-plane.

The proof relies on the classical vectorfield method and bootstrapping argument from Christodoulou-Klainerman. The main novelty is the introduction and control of new geometric constructions adapted to the spacelike-characteristic setting. In particular, it features null cones with prescribed vertices, spacelike maximal hypersurfaces with prescribed boundaries and global harmonic coordinates on Riemannian 3-disks.

Résumé (Stabilité globale non-linéaire de l'espace de Minkowski pour des données initiales spatiales-caractéristiques)

Dans cet article nous prouvons la stabilité globale non-linéaire de l'espace de Minkowski dans le cadre du problème de Cauchy spatial-caractéristique pour les équations d'Einstein dans le vide. Les données initiales spatiales-caractéristiques sont posées sur un 3-disque et sur l'hypersurface nulle et complète dans le futur qui émane du bord de ce disque. Notre résultat étend le résultat originel prouvé par Christodoulou et Klainerman pour lequel les données initiales sont prescrites sur un hyperplan spatial.

La preuve repose sur la méthode des champs de vecteurs et sur l'argument de bootstrap introduit dans Christodoulou-Klainerman. La principale nouveauté est l'introduction et le contrôle de nouvelles constructions géométriques adaptées au cadre spatial-caractéristique. En particulier, on utilise des cônes de lumière à sommets prescrits, des hypersurfaces spatiales maximales à bords prescrits et des coordonnées harmoniques globales sur des 3-disques riemanniens.

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