

GROUPS ACTING ON FRACTALS, HYPERBOLICITY AND SELF-SIMILARITY

François Dahmani, Anna Erschler,
Camille Horbez, Daniel Wise (eds.)



Panoramas et Synthèses

Numéro 63

2025

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Comité de rédaction

Samuel BOISSIÈRE	Diego IZQUIERDO
Fabienne CASTELL	Claire LACOUR
Indira CHATTERJI	Quentin MÉRIGOT
Élise GOUJARD	Sergio SIMONELLA
Anne-Sophie de SUZZONI	Todor TSANKOV
Anne MOREAU (dir.)	

Diffusion

Maison de la SMF	AMS
Case 916 - Luminy	P.O. Box 6248
13288 Marseille Cedex 9	Providence RI 02940
France	USA
christian.smf@cirm-math.fr	www.ams.org

Tarifs

Vente au numéro : 59 € (\$ 89)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat

Panoramas et Synthèses
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
panoramas@smf.emath.fr • <http://smf.emath.fr/>

© Société Mathématique de France 2025

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN 1272-3835

ISBN 978-2-37905-215-6

Directrice de la publication : Isabelle Gallagher

PANORAMAS ET SYNTHÈSES 63

**GROUPS ACTING ON FRACTALS, HYPERBOLICITY
AND SELF-SIMILARITY**

**François Dahmani, Anna Erschler,
Camille Horbez & Daniel Wise (eds.)**

Société mathématique de France

François Dahmani

Institut Fourier, UMR 5582, Laboratoire de Mathématiques Université Grenoble Alpes, CS 40700, 38058 Grenoble Cedex 9, France

E-mail : francois.dahmani@univ-grenoble-alpes.fr

ORCID : 0000-0003-3874-7436

Anna Erschler

Sorbonne Université. LPSM. 4 place de Jussieu, 75252 Paris Cedex 5, France

E-mail : anna.erschler@sorbonne.universite.fr

ORCID : 0000-0002-7510-9890

Camille Horbez

Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, CNRS, Batiment 307 91405 Orsay Cedex, France

E-mail : camille.horbez@cnrs.fr

ORCID : 0000-0003-1933-445X

Daniel Wise

Dept. of Mathematics Statistics, Burnside Hall, McGill University, 805 Sherbrooke Street West, Montreal, Quebec, Canada

E-mail : wise@math.mcgill.ca

ORCID : 0000-0003-0128-1353

Classification mathématique par sujets. (2020) — 20Fxx, 20Exx, 37D20, 60G50, 28A80, 28A78, 37F35.

Keywords and phrases. — Geometric group theory, dynamics on groups, growth, fractalness, self-similar groups, automorphism groups of free groups and surface groups, hyperbolic groups, boundaries of groups, groups acting on trees.

Mots-clés et phrases. — Théorie géométrique des groupes, dynamique sur les groupes, croissance, fractalité, groupes auto-similaires, groupes d'automorphismes de groupes libres et de groupes de surfaces, groupes hyperboliques, bords de groupes, groupes agissant sur des arbres.

GROUPS ACTING ON FRACTALS, HYPERBOLICITY AND SELF-SIMILARITY

François Dahmani, Anna Erschler,
Camille Horbez & Daniel Wise (eds.)

Abstract. — This volume collects the lecture notes of the mini-courses given during the special trimester *Groups Acting on Fractals, Hyperbolicity and Self-Similarity*, held at the Institut Henri Poincaré in April-July 2022. Its guiding principle was fractalness in group theory, articulated around three main topics: self-similar groups acting on rooted trees, hyperbolic groups and their boundaries, and automorphism groups of free groups and surface groups. We hope this collection of mini-courses will highlight the rich and fruitful interactions between these topics, and provide a resource for a renewed community of researchers in the field.

Résumé. (Groupes agissant sur des fractals, hyperbolicité et auto-similarité) — Ce volume rassemble les notes des mini-cours donnés lors du trimestre spécial *Groupes agissant sur des fractales, hyperbolicité et auto-similarité*, qui s'est tenu à l'Institut Henri Poincaré d'avril à juillet 2022. Son principe directeur était la fractalité dans les groupes, articulée autour de trois principaux thèmes : les groupes auto-similaires agissant sur des arbres enracinés, les groupes hyperboliques et leurs bords, et les automorphismes de groupes libres et de groupes de surfaces. Nous espérons que ce recueil de mini-cours soulignera les interactions riches et fructueuses entre ces thèmes, et constituera une ressource pour une communauté sans cesse renouvelée de personnes travaillant dans ce domaine.

TABLE DES MATIÈRES

Table des matières	vii
Abstracts	ix
Foreword	xi
RACHEL SKIPPER — <i>Groups acting on rooted trees</i>	1
1. The (first) Grigorchuk Group	2
2. Self-similar groups and virtual endomorphisms	9
3. Automata groups	11
4. Branch groups and the classification of just infinite groups	15
References	17
VOLODYMYR NEKRASHEVYCH — <i>Hyperbolic groupoids</i>	19
1. Introduction	19
2. Contracting groups	21
3. Limit solenoid and its leaves	26
4. Ruelle-Smale systems	28
5. Hyperbolic groupoids	33
References	47
TIANYI ZHENG — <i>Random walks on Grigorchuk groups and growth estimates</i>	49
1. Introduction	49
2. Isoperimetry	54
3. Random walks on Grigorchuk groups	58
References	66
GEMMA CROWE, PAIGE HELMS & KAREN VOGTMANN — <i>The geometry and topology of automorphism groups of free groups</i>	69
1. Introduction	69
2. Graphs and Outer space	70

3. Trees in Outer space	74
4. Spheres and Outer space	80
5. Related groups and spaces	84
6. Homology	86
References	90
MLADEN BESTVINA & SOUMYA DEY — <i>Projection complexes and applications to mapping class groups</i>	
	93
1. Introduction	93
2. Construction of Projection Complex	94
3. Construction of quasi-tree of metric spaces $\mathcal{C}_K(\mathcal{Y})$	105
4. Applications to Mapping Class Groups	109
References	113
RÉMI COULON & ALEXIS MARCHAND — <i>Growth and co-growth in negatively curved groups: a dynamical point of view</i>	
	115
Introduction	115
1. Background on CAT(−1) spaces	119
2. Conformal densities	121
3. The Patterson construction	123
4. Quasi-convex subgroups	125
5. The Sullivan Shadow Lemma	127
6. Growth rate for normal subgroups	130
7. Co-amenability criterion	134
8. Beyond hyperbolic groups	138
References	141
JOHN M. MACKAY — <i>Conformal dimension and hyperbolic groups</i>	
	145
1. What is conformal dimension?	145
2. Splittings and conformal dimension	154
3. Random groups and conformal dimension	159
4. Coarse embeddings and conformal dimension	165
References	168
ANTHONY GENEVOIS — <i>Mini-course on the embedding problem between RAAGs</i>	
	171
1. Lecture 1: A crash course on RAAGs	171
Exercises	176
2. Lecture 2: A crash course on (quasi-)median graphs	177
3. Lecture 3: Creating RAAGs as subgroups	184
Exercises	187
4. Lecture 4: An algorithm in dimension two	188
Exercises	191
References	192

ABSTRACTS

<i>Groups acting on rooted trees</i> RACHEL SKIPPER	1
<i>Hyperbolic groupoids</i> VOLODYMYR NEKRASHEVYCH	19
<i>Random walks on Grigorchuk groups and growth estimates</i> TIANYI ZHENG	49
These are notes for the mini-course with the same title given in the thematic trimester <i>Groups acting on fractals: hyperbolicity and self-similarity</i> , at Institut Henri Poincaré.	
<i>The geometry and topology of automorphism groups of free groups</i> GEMMA CROWE, PAIGE HELMS & KAREN VOGTMANN	69
This article is based on lectures delivered by the third author virtually at CIRM in Luminy, as part of the 2022 IHP trimester ‘Groups acting on fractals, hyperbolicity and self-similarity’, from notes taken by the first two authors.	
<i>Projection complexes and applications to mapping class groups</i> MLADEN BESTVINA & SOUMYA DEY	93
This article is based on lectures delivered by the first author at the Institut Henri Poincaré, Paris as part of the thematic trimester titled ‘Groups acting on fractals, hyperbolicity and self-similarity’ held during April-July, 2022, from notes taken by the second author.	
<i>Growth and co-growth in negatively curved groups: a dynamical point of view</i> RÉMI COULON & ALEXIS MARCHAND	115
This article is based on lectures delivered by the first author at CIRM in Luminy, as part of the 2022 IHP trimester ‘Groups acting on fractals, hyperbolicity and self- similarity’, from notes taken by the second author.	
<i>Conformal dimension and hyperbolic groups</i> JOHN M. MACKAY	145
These notes are based on a minicourse given in June 2022 at Institut Henri Poincaré, Paris as part of the thematic trimester program <i>Groups acting on fractals, hyperbolicity and self-similarity</i> .	

Mini-course on the embedding problem between RAAGs

ANTHONY GENEVOIS 171

Given a graph Γ , the *right-angled Artin group* $A(\Gamma)$, also referred to as a *partially commutative group*, is given by the presentation

$$\langle u \in V(\Gamma) \mid [u, v] = 1, \{u, v\} \in E(\Gamma) \rangle.$$

This mini-course is dedicated to the Embedding Problem in right-angled Artin groups, namely: given two finite graphs Φ, Ψ , how to determine whether or not $A(\Phi)$ is isomorphic to a subgroup of $A(\Psi)$? We describe a geometric framework, based on the geometry of *quasi-median graphs*, in which this question can be studied.