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HERMITE'S "CONCRETE" ANALYSIS: RESEARCH AND EDUCATIONAL THEMES IN AN EVOLVING DISCIPLINE

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Abstract. — The emerging discipline of mathematical analysis exhibited various threads during the nineteenth century, with different values and priorities as to basic definitions and approaches being used in different national and local contexts. In this paper we examine the "concrete" analysis of Charles Hermite, looking at its roots in his own research and the developing pedagogical versions of it that appeared in his courses and the work of certain of his students.

Résumé (L'analyse « concrète » de Hermite : recherche et thématiques éducationnelles dans une discipline en évolution)

La discipline émergente d'analyse mathématique manifestait diverses versions au long du XIX^e siècle, avec différentes communautés locales adoptant des valeurs et priorités différents dans leurs choix de définitions et d'approches. Dans cet article on focalise sur l'analyse « concrète » de Charles Hermite, regardant les racines de son approche dans ses propres recherches, ainsi que les versions pédagogiques qui ont paru dans ses cours et dans les travaux de ses étudiants.

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1. INTRODUCTION

A great deal of the work of Charles Hermite falls under what we would now term analysis. As Catherine Goldstein and Norbert Schappacher argue in their landmark study of the legacy of Gauss, much of his work may reasonably be termed arithmetic algebraic analysis [Goldstein et al. 2007]. The term algebraic analysis was used already in the eighteenth century, and continues to be used by historians, to describe a certain body of work, vague in extent, that concerns itself with infinite and infinitesimal mathematics and its applications (for example in differential equations) while using techniques that even today we would naively term algebraic, focussing on issues having to do with symbol manipulation. Such work may indirectly address concerns that we would term geometric (that is, involving curves and surfaces) or even analytic (involving estimation and convergence questions) but this is not its primary focus [Fraser 1989; Jahnke 2003]. I will not dwell on the "arithmetic" label here, except to note that this refers to applications in number theory. In Hermite's case, this grew directly out of the work of Jacobi, shown in his early publications and in his correspondence with the latter [Hermite 1850b]. Hermite's work in number theory has been discussed in detail in [Goldstein 2007] and [Goldstein 2011a], where Hermite's views about the relation of this work to analysis are explored.

This focus was the starting point of a long career in which Hermite worked not only in this area, but in questions related to elliptic functions more generally, especially focussing on concrete representations of these, often in terms of theta functions. Other work of Hermite may be reasonably classed with the "analysis" label, notably involving differential equations and one famous result, the transcendence of e. Our purpose in this paper is not to survey all aspects of this work, but rather to sample it, looking at the features of what Hermite himself termed analysis, which is certainly not the analysis of today. Nor is it the analysis of Augustin-Louis Cauchy or Leonhard Euler, despite a certain backward-looking character. Our sample will be somewhat opportunistic, but will also include some discussion of the introductory teaching of the subject, pointing to various innovations Hermite attempted to introduce. We will also look very briefly at the work of some of the doctoral students he mentored, and comment on their role in shaping education in this area. Some of his research students taught in preparatory schools, a uniquely French form that emerged in the mid-nineteenth century with the particular aim of coaching students to succeed in entrance examinations to the most important of the state scientific or technical schools.

In what follows we will first give some examples of his practice. We then turn to some specific writings aiming at transmitting important parts of that practice: the appendix to the 1862 edition of Lacroix's *Calcul* on elliptic functions; the *Cours d'analyse* given at the École polytechnique in 1872, and the 1882 lectures at the Faculté des sciences of the Sorbonne, also referred to as a *Cours d'analyse* but very different from the earlier work. We will then look at three of the thesis student who had specific teaching roles. It is difficult, often, to determine which students had a thesis that might be thought of as directed by Hermite, but at least a dozen either acknowledge him directly or work on themes directly related to his own work.

In so doing, we want to point out also that the nineteenth century reforms in the foundations of analysis, for example those of Cauchy and Weierstrass, in no way led to the immediate extinction of all research in the older, more concrete and formula-based vein that had its origin in the eighteenth century. Consider a statement such as the one due to Giovanni Ferraro:

> [In the early 1820s] Cauchy published *Cours d'analyse* and *Résumé des leçons données à l'École Royale Polytechnique sur le calcul infinitésimal*, which can be considered to mark the definitive abandonment of the eighteenth century formal approach to series theory. [Ferraro 2008, vii]

Not to blame Ferraro unfairly, since the next period was not his subject, but such casual remarks often make it seem that the innovations of Cauchy possessed the kind of rapid revolutionary effect that the statement seems to suggest. In what follows, we will see that in fact this older algebraic analysis lived on and thrived, in an influential and important Hermitian form that remained, and remains, part of the developing world of analysis.

In looking at aspects of the evolution of analysis, we may consider this label as describing a discipline, field, or research specialty, Distinguishing between these terms is not something we will attempt here in detail, but we note a few features. The idea of a discipline is one that has developed over time. In Aristotle, for example, the term is associated with something that must be learned, and in the Aristotelian context this meant that it was associated with a particular subject. By about 1900, the idea enters into sociology, so that Max Weber associates it with professional practice, and with the idea of how to reproduce a given set of practices. Newer sociological versions, for example by Pierre Bourdieu, retain this while analysing further the social character of what is being transmitted, refining ideas about the fundamental entity and attempting to cut through some confusions via the notion of a field. Our usage will concentrate on the basic feature of transmission of knowledge and approach.

In mathematics, the term discipline is used somewhat vaguely to distinguish what are often termed branches of the subject—things like algebraic topology or finite geometries. These remain associated with research specialties, and hence with practice both in research and teaching. One can thus look at a body of mathematical work as being associated with one or several fields of activity. From the viewpoint of the mathematical researcher (and trainer of researchers) this brings to the fore how to acquire the techniques of practice. The meaning of discipline in the context of the history of mathematics, and its relation to ideas of specialty or of mathematical practice, are examined in several works by Sébastien Gauthier, notably [Gauthier 2007] and [Gauthier 2009]. In the context of number theory, it is discussed in relation to the work of Gauss by Catherine Goldstein and Norbert Schappacher in the first two chapters of [Goldstein et al. 2007].

Analysis, rather than a discipline or specialty, was originally a method, the opposite of synthesis, and hence part of a practice of proof, or of description of proof. Algebra is an analytical method in that antique sense, and the addition of differential and integral techniques enriched the set of algebraic methods beginning (in Europe) in the seventeenth century. This set of novelties led to a body of work that supplemented older algebra and led to a new set of problems that could be attacked using these methods. Over the course of the nineteenth and twentieth centuries, this somewhat fluid picture hardened into a more or less codified set of practices that were to be mastered by the disciples.

2. HERMITE'S STARTING POINT: ANALYSIS IN FRANCE CIRCA 1840

As already observed, the older algebraic analysis emphasized formulas and symbol manipulation as a way to produce and control them. Already by the early nineteenth century, working with formulas had led to various paradoxical results, as is well known, and Hermite's mentor Cauchy was famously critical of what he had termed the "generalities of algebra," an approach from which he disassociated his own work in the Introduction to his *Analyse Algébrique* [Gilain 1989]. It is noteworthy, though, that Cauchy's own foundational efforts involving limits did not at once find general use. Nor, when used, did they provide a clear way to resolve many issues, notably in cases involving multiple limits at once. Similarly, despite the clarifications provided by Cauchy in the field of complex function theory, the uptake was slow [Bottazzini & Gray 2013, chs 2 and 3].

Hence the older, eighteenth-century forms held sway until at least 1850 in many contexts, both in France and elsewhere. Hermite, a "little Lagrange" in the words of his lycée mathematics professor Louis Paul Émile Richard, started research while at the École polytechnique [Hermite 1905–1917, vii, Préface by E. Picard]. The work of Lagrange and Abel were his main starting points, to judge from his early publications. His first good results built on work of C. G. J. Jacobi, and Hermite was encouraged by Joseph Liouville, an important mentor and supporter, to send these to Jacobi. These concerned the division of elliptic functions and the extension of some of Jacobi's results to hyperelliptic functions. Jacobi's positive response was read aloud at the Académie. To give some impression of context for at least some of Hermite's interests, we summarize some work of Jacobi.

The Context of Hermite's Early Work: Jacobi and Abel

Jacobi, in the 1829 *Fundamenta*, recalled the theorem of Euler, that if

$$\Pi(x) = \int_0^x \frac{dx}{\sqrt{X}},$$

where X is a polynomial of degree four, then we may write

$$\Pi(x) + \Pi(y) = \Pi(a),$$

where a is an algebraic function of x and y. [Jacobi 1829]

Niels Henrik Abel had extended this to polynomials of any degree. The more general result, called Abel's theorem by Jacobi, states:

Let X be a polynomial of degree $2m$ or $2m-1$, and define

$$\Pi(x) = \int_0^x \frac{(A + A_1 x + \cdots + A_{m-2}x^{m-2})\,dx}{\sqrt{X}},$$

then given m values of the variable x it is possible to determine from them $m-1$ quantities a_i such that

$$\Pi(x) + \Pi(x_1) + \cdots + \Pi(x_{m-1}) = \Pi(a) + \Pi(a_1) + \Pi(a_{m-2}).$$

An open question concerned what the inverses of the Abelian integrals are like, and what Abel's theorem can tell us about them.

Given our interest in Hermite's style of analysis, we note that his immersion in Jacobi's work must have influenced him considerably. Jacobi, emphasizing the production of formulas as a key part of investigating the properties of the functions whose analysis concerned him, frequently took what might be termed an algebraic point of view. He stressed for example