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A CONVERGENCE OF PATHS: CAYLEY, HERMITE, SYLVESTER, AND EARLY INVARIANT THEORY

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In loving memory of Brian J. Parshall
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Abstract. — This paper considers the beginnings of a *theory* of invariants in the early 1850s in the broader contexts of individual pathways toward the establishment of reputation and of the professionalization of mathematics in the nineteenth century. In particular, it treats the different, but intersecting, mathematical paths by which two Englishmen, Arthur Cayley and James Joseph Sylvester, and one Frenchman, Charles Hermite, came to focus on an analysis *per se* of the transformation of homogeneous forms by linear substitutions. It then looks at the intense mathematical exchanges in the first half of the 1850s that resulted in their early invariant-theoretic results. Although by the close of the 1850s, Cayley, Hermite, and Sylvester had largely gone their own separate mathematical ways, the three remained united in their sense of having created what they called the "New Algebra."

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Mots clefs. — Arthur Cayley, Charles Hermite, James Joseph Sylvester, history of invariant theory, nineteenth century, professionalization and internationalization of mathematics.

Résumé. — Cet article considère les origines d'une *théorie* d'invariants au début des années 1850 dans les contextes plus larges des chemins suivis pour établir une réputation mathématique et de la professionnalisation des mathématiques au XIXe siècle. En particulier, il s'occupe des différentes voies mathématiques, mais néanmoins voies croisées, par lesquelles deux Anglais, Arthur Cayley et James Joseph Sylvester, et un Français, Charles Hermite, en vinrent à se concentrer sur une analyse de la transformation de formes homogènes par les substitutions linéaires. Il se penche ensuite sur les échanges mathématiques intenses dans la première moitié des années 1850, échanges qui ont abouti aux premiers résultats proprement dits invariant-théoriques. Bien qu'à la fin des années 1850, les chemins mathématiques de Cayley, Hermite, et Sylvester avaient largement divergés, les trois mathématiciens sont restés unis dans le sentiment d'avoir créé ce qu'ils ont appellé la « nouvelle algèbre. »

Arthur Cayley was a twenty-two-year-old assistant tutor at and fellow of Trinity College, Cambridge in June 1844 when he wrote to George Boole, a Lincolnshire schoolteacher more than six year's his senior. The 1842 Cambridge Senior Wrangler had been reading a paper published by the largely self-taught Boole in the *Cambridge Mathematical Journal* and had produced "a few formulae relative to it" that he hoped would spark Boole's interest. [1] Following on his reading primarily of Joseph-Louis Lagrange's *Méchanique analitique* and the *Mécanique céleste* of Pierre Simon de Laplace, Boole had been intrigued by what he styled an "important and oft recurring problem of analysis," namely, "[t]he transformation of homogeneous functions by linear substitutions" [Boole 1841-1843a, p. 1]. Cayley, spurred to a large extent by "analytical geometry, his growing passion in mathematics" [Crilly 2006, p. 86], carried Boole's ideas further with the "formulae" that he published in 1845 and that marked, in some sense, his entrée into what would later become a theory of invariants [Cayley 1845].

Instances of the transformation of homogeneous functions by linear substitutions also cropped up in settings other than the mathematization of mechanics. Carl Friedrich Gauss had explored, in the number-theoretic context of his *Disquisitiones arithmeticae* of 1801, the question of how a binary quadratic form with integer coefficients was affected by a linear transformation [Gauss 1966, pp. 111-112]. It was Charles Hermite's independent reading of that source, as well as of Lagrange's *Traité de la résolution des équations numériques de tous les degrés* (first published in 1789 and revised by the author in 1808), that had exposed the *collégien* to both higher algebra and number theory while at Paris's Collège Louis-le-Grand

[1] The letter is quoted, among other places, in [Crilly 2006, p. 86] and [Wolfson 2008, p. 43].

in the early 1840s [Picard 1905, p. viii]. By 1848, Hermite, who was one year Cayley's junior, had passed his *baccalauréat* and *licence*, had become a *répétiteur* and admissions examiner at the École polytechnique, and had published a note in which he, too, focused on the problem of transformation, but from a number-theoretic point of view [Hermite 1848].

A year earlier, yet another mathematician, James Joseph Sylvester, had also come to the matter of transformation from number theory [Sylvester 1847a], although he had cut his mathematical eyeteeth in the 1830s on a purely algebraic approach to the theory of elimination, that is, the theory involved, for example, in finding when two polynomial equations of degrees m and n in one variable have a common root or in determining the real roots of an algebraic equation $f(x) = 0$ of degree n. [2] Almost seven years Cayley's senior, Sylvester, a Jew and the Second Wrangler in 1837, had had a checkered career as he had tried, ultimately unsuccessfully, to establish himself as a mathematician in academe both in England and the United States over the course of the late 1830s and early 1840s. Back in England by the close of 1843, he took a position in 1844 as an actuary at the Equity and Law Life Assurance Society in London. This soon put him in close proximity to Cayley, who had left Cambridge in 1846 to prepare for a career at the Bar. By 1847, the two had met and started up what would become the mathematical correspondence they would maintain for essentially the rest of their lives. [3]

It is not clear exactly when Sylvester and Cayley met Hermite. Both Englishmen were, however, intent on making mathematical reputations for themselves in England and beyond. They had both participated in the French mathematical scene in the 1840s, and both would publish regularly in European journals. It is clear that Sylvester united the three of them in print in 1851 under the common rubric of "transformation" in his paper, "Sketch of a Memoir on Elimination, Transformation, and Canonical Forms" [Sylvester 1851e]. There, he gave an early statement of "invariance" as it had emerged in the work of Boole and Cayley, at the same time that he referred to his "admirable friend M. Hermite" [Sylvester 1851e, pp. 185 and 190, resp.]. [4] Over the course of the first half of the 1850s, these three young mathematicians—separated by the English Channel—made common cause in the development of a new

2 See the discussion in [Parshall 2006, pp. 59-62].

3 This early period in Sylvester's life is treated in [Parshall 2006, pp. 49-94]. For a glimpse of Cayley and Sylvester's correspondence, see [Parshall 1998].

4 Page references given for papers by Sylvester, Cayley, and Hermite in what follows refer to the pagination in their respective collected works.

theory—the theory of invariants—or what they would unabashedly term "the New Algebra" [Sylvester 1851a, p. 252]. At the same time, they worked to establish their respective careers in mathematics.

Although much has been written on Cayley and Sylvester's roles in the early development of invariant theory,[5] neither Hermite's part in that development nor his relationship with Cayley and Sylvester and with what became a British school of invariant theory has received particular historical scrutiny. What mathematical paths led them to focus on an analysis *per se* of the transformation of homogeneous forms by linear substitutions? What was the dynamic of the mathematical interchange between them—two in England and one in France—in the first half of the 1850s that resulted in their early invariant-theoretic results? In addressing these questions, this paper not only highlights Hermite's participation in the early development of what was later recognized as the British strain of invariant theory but also provides an interesting case study of how new mathematical ideas could develop in the mid-nineteenth century.

CAYLEY'S ANALYTICAL-GEOMETRICAL PATH

Boole's 1841 paper, "Exposition of a General Theory of Linear Transformations," focused on the determination of "the relations by which" the coefficients of a homogeneous polynomial of degree n in m unknowns "are held in mutual dependence" before and after a linear transformation of its variables is applied.[6] The general technique that he developed to address this problem involved the elimination of the variables from the given polynomial via partial differentiation with respect to each of its unknowns, and he illustrated it in a number of specific examples before stating a general result.

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5 See, for example, [Crilly 1986], [Parshall 1989], [Parshall 1998], [Crilly 2006], and [Parshall 2006]. The latter three works also provide details on the subsequent evolution of invariant theory at the hands of Cayley and Sylvester as does Crilly [1988], while Parshall [1989] analyzes and compares the contemporaneous British and German approaches to the field. And, although David Hilbert purportedly struck a near fatal blow to invariant theory in 1890 with the publication of his paper "Über die Theorie der algebraischen Formen" [Hilbert 1890] and actually "killed" it in 1893 [Hilbert 1893], sociologist of science Charles Fischer pronounced the field "dead" only by the 1920s in [Fischer 1966] and [Fischer 1967]. See [Parshall 1990], however, for more on the "death" argument.

6 See [Boole 1841-1843a, p. 3], as quoted in [Parshall 1989, p. 161]. In what follows, I have massaged the notation in the original papers in order to generate a notation for this paper more consistent across the various works discussed.

Consider, as did Boole, the simplest case of the binary quadratic form

$$Q = ax^2 + 2bxy + cy^2, \tag{1}$$

that is, the homogeneous polynomial of degree two in two unknowns, where a, b, and c are implicitly real numbers [Boole 1841-1843a, p. 6]. Calculating the partial derivatives of Q with respect to x and y and setting the results equal to zero generated two equations,

$$\frac{\partial Q}{\partial x} = 2ax + 2by = 0$$
$$\frac{\partial Q}{\partial y} = 2bx + 2cy = 0,$$

from which Boole eliminated the variables to get the expression

$$\theta(Q) = b^2 - ac,$$

that is, one of the desired relations by which the coefficients of Q "are held in mutual dependence" or what was later termed by Sylvester the discriminant of Q.[7] He next applied the linear transformation

$$\begin{aligned} x &= mx' + ny' \\ y &= m'x' + n'y', \end{aligned} \tag{2}$$

for m, n, m', $n' \in \mathbb{R}$ (and $mn' - m'n \neq 0$, although he assumed, but did not explicitly note, this restriction) to Q to get a new binary quadratic form

$$R = A(x')^2 + 2Bx'y' + C(y')^2.$$

Clearly, calculating the partial derivatives of R yielded

$$\theta(R) = B^2 - AC.$$

Later in his paper [Boole 1841-1843a, p. 19], Boole proved, by explicit calculation, that $\theta(R)$ and $\theta(Q)$ were equal up to a power of the determinant of the linear transformation (2).

Repeating the same analysis *mutatis mutandis* for the binary cubic $Q = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ shows that its discriminant

$$(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)$$

also remains unchanged up to a power of the determinant of the linear transformation. In language that would emerge only later (see below), but

[7] Sylvester coined the term "discriminant" in a letter to Cayley dated 25 August, 1851. As he put it, the purpose of that letter was to "submit" for Cayley's "approval & ratification" a number of terms in addition to "discriminant," among them, "invariant" and "resultant." See [Parshall 1998, pp. 35-37]. Sylvester used the term "discriminant" in print for the first time in [Sylvester 1851d, p. 280].