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CONGRUENCES OF MODULAR FORMS AND THE IWASAWA λ-INVARIANTS

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ABSTRACT. — In this paper, we show how congruences between cusp forms and Eisenstein series of weight $k \geq 2$ give rise to corresponding congruences between the algebraic parts of the critical values of the associated L-functions. This is a generalization of results of Mazur, Stevens, and Vatsal in the case where $k = 2$. As an application, by proving congruences between the p-adic L-function of a certain cusp form and the product of two Kubota-Leopoldt p-adic L-functions, we prove the Iwasawa main conjecture (up to p-power) for cusp forms at ordinary primes p when the associated residual Galois representations are reducible. This is a generalization of Greenberg and Vatsal in the case where $k = 2$.

RÉSUMÉ (*Congruences de formes modulaires et λ-invariants d'Iwasawa*). — Dans cet article, nous montrons comment les congruences entre formes paraboliques et séries d'Eisenstein de poids $k \geq 2$ donnent lieu à des congruences entre les parties algébriques des valeurs critiques des fonctions L associées. C'est une généralisation des travaux de Mazur, Stevens et Vatsal dans le cas où $k = 2$. Comme application, en prouvant des congruences entre la fonction p-adique L d'une certaine forme parabolique et le produit de deux fonctions de Kubota-Leopoldt p-adiques L, nous prouvons la conjecture principale d'Iwasawa (à puissance p près) pour les formes paraboliques à nombres premiers ordinaires p lorsque les représentations de Galois résiduelles associées sont réductibles. C'est une généralisation des travaux de Greenberg et Vatsal dans le cas où $k = 2$.

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0. Introduction

0.1. Introduction. — The purpose of this paper is to show how congruences between the Fourier coefficients of Hecke eigenforms give rise to corresponding congruences between the special values of the associated L-functions. The study of this topic was initiated by Mazur [25] using the arithmetic of the modular curve $X_0(l)$, where l is a prime number, in order to investigate a weak analog of the Birch and Swinnerton-Dyer conjecture. Mazur's congruence formula was generalized to other congruence subgroups by Stevens [33]. Furthermore, by the theory of higher weight modular symbols, Ash and Stevens [2] have examined congruences between special values of the L-functions of cusp forms of higher weight over $\mathrm{SL}_2(\mathbb{Z})$ and those of the L-functions of cusp forms of weight 2 over $\Gamma_0(l)$. Moreover, Vatsal [39] has proved congruences between special values of the L-functions of two cusp forms of higher weight over $\Gamma_0(N)$, where N is a more general positive integer. Also, he obtained congruences between special values of the L-functions of cusp forms of weight 2 and those of the L-functions of Eisenstein series of weight 2. Moreover, Greenberg and Vatsal [16] used Vatsal's congruences [39] to study the Iwasawa invariants of elliptic curves in towers of cyclotomic fields. In particular, they provided evidence for the Iwasawa main conjecture for elliptic curves. Their work was motivated by Kato's results on the Iwasawa main conjecture for modular forms [21].

In this paper, we present a way to obtain congruences of the special values of the L-functions from congruences between cusp forms and Eisenstein series of weight $k \geq 2$. This is a generalization of the works explained above by Mazur [25], Stevens [33], and Vatsal [39].

Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$ and $\varpi \in \mathcal{O}$ a uniformizer.

THEOREM 0.1 (= Theorem 2.10). — *Let p be an odd prime number, r a positive integer, and k an integer with $2 \leq k \leq p-1$. Let $f = \sum_{n=1}^{\infty} a(n,f)\ e(nz) \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a p-ordinary normalized Hecke eigenform. Assume that the residual Galois representation $\bar{\rho}_f$ associated to f is reducible and of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix},$$

and either ξ_1 or ξ_2 is unramified at p. Assume also that there exists an Eisenstein series $G = E_k(\psi_1, \psi_2) \in M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ (for the definition, see Theorem 3.18) such that G satisfies the assumptions of Theorem 1.9 and $f \equiv G \pmod{\varpi^r}$ (for the definition, see before Theorem 2.10). Then there exist a parity $\alpha \in \{\pm 1\}$ (explicitly given by (A.27)), a complex number $\Omega_f^{\alpha} \in \mathbb{C}^{\times}$, and a p-adic unit $u \in \mathcal{O}^{\times}$ such that, for every primitive Dirichlet character χ whose conductor m_χ is prime to N, the following congruence holds:

(1) *if* $(m_\chi, p) = 1$, *then, for each* j *with* $0 \le j \le k-2$ *and* $\alpha = \chi(-1)(-1)^j$,

$$\tau(\bar{\chi})\frac{L(f,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}\Omega_f^\alpha} \equiv u\tau(\bar{\chi})\frac{L(G,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}} \pmod{\varpi^r}.$$

(2) *if* $p|m_\chi$, *we assume that* $m_\chi \in \varpi^r\mathcal{O}$, χ *is non-exceptional (see Definition 2.11), and* $\alpha = \chi(-1)$. *Then*

$$\tau(\bar{\chi})\frac{L(f,\chi,1)}{(2\pi\sqrt{-1})\Omega_f^\alpha} \equiv u\tau(\bar{\chi})\frac{L(G,\chi,1)}{2\pi\sqrt{-1}} \pmod{\varpi^r}.$$

The organization of this paper is as follows.

In §1, we generalize Stevens's results [33, 34]. We construct a desired 1-cocycle π_g associated to a modular form g of weight $k \ge 2$ (Definition 1.2) and prove that π_g is integral, that is, π_g takes values in $L_{k-2}(\mathcal{O})$ under some assumption (Theorem 1.9). In terms of Schoenberg's cocycle, Stevens gave a generalization of the Mazur's congruence formula [25] to general congruence subgroups [33]. Also, he expected that these methods would be generalized to higher weight modular forms and to Hilbert modular forms [33]. The construction of such cocycles π_g associated to modular forms g of weight k has been accomplished so far only in the case of weight $k = 2$ mainly because of certain combinatorial problem arising in the higher weight case $k > 2$. Indeed, a discrete subgroup Γ acts on $L_{k-2}(\mathcal{O})$ trivially only in the case $k = 2$.

In §2, we generalize Vatsal's results [39].

If a Hecke eigenform $f = \sum_{n=1}^{\infty} a(n,f)e(nz)$ of weight $k \ge 2$ and an Eisenstein series $G = \sum_{n=0}^{\infty} a(n,G)e(nz)$ of weight $k \ge 2$ are related by a congruence of the Fourier coefficients $a(n,f) \equiv a(n,G) (\mathrm{mod}\ \varpi^r)$ for all $n \ge 0$, we derive congruences between the special values of the associated L-functions (Theorem 2.10). One of the key ingredients in Vatsal's proof [39] is to describe the special values of the L-functions attached to the modular form G as a linear combination of 1-cocycles π_G due to the work of Stevens [33], which allows us to prove congruences between the special values by using cohomological arguments.

In Appendix A, we give a relation between p-adic modular forms and p-adic parabolic cohomologies of Hecke modules in the case the residual Galois representations $\bar{\rho}_f (= \rho_f (\mathrm{mod}\ \varpi))$ associated to a cusp forms f is reducible by using integral p-adic Hodge theory. Our problem on the special values of the L-functions is closely related to a multiplicity-one theorem, which is introduced by Mazur. In the case $\bar{\rho}_f$ is irreducible, $k < p$, and a level N is prime to p, a multiplicity-one theorem is known to be valid by p-adic Hodge theory for open varieties with non-constant coefficients [10]. In particular, Theorem A.12, which may be regarded as p-adic Eichler-Shimura isomorphism, is crucial to define the canonical periods Ω_f^α associated to f and prove congruences between $\pi_f^\alpha/\Omega_f^\alpha$ and π_G^α modulo ϖ^r.

In §3, we generalize Greenberg-Vatsal's results [16]. Using Vatsal's congruences, it is devoted to an application to the Iwasawa main conjecture for elliptic curves under certain assumptions. In the same manner, Theorem 0.1 is used to establish a congruence between a p-adic L-function attached to f and the product of two Kubota-Leopoldt p-adic L-functions (Theorem 3.19). Then, following the work of Kato [21], we will prove the following theorem, which has not been treated by Skinner and Urban [32]:

THEOREM 0.2. — *Let p be an odd prime number and k an integer such that $2 \leq k \leq p-1$. Let $f \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a p-ordinary normalized Hecke eigenform. We assume that the residual Galois representation $\bar{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\varpi)$ associated to f is reducible and of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix}$$

and that

$$\text{(Assumption)} \qquad \begin{aligned} &\psi \text{ is unramified at } p \text{ and odd, and} \\ &\varphi \text{ is ramified at } p \text{ and even.} \end{aligned}$$

Then $\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}$. In particular, the Iwasawa main conjecture for such f is true up to ϖ-power.

The work of §1, §2, and §3 is based on the author's master thesis at the University of Tokyo in 2010. After I had finished writing this paper, I found a result obtained by Heumann and Vatsal [17], which is almost the same one as Theorem 0.1 (1) (in the case $(m_\chi, p) = 1$) in this paper. We also treat the case $p | m_\chi$ (Theorem 0.1 (2)) and apply Theorem 0.1 (2) to the Iwasawa main conjecture.

0.2. Notation. — In this paper, p and l always denote distinct prime numbers.

We denote by $\mathbb{N}$ the set of natural numbers (that is, positive integers), denote by $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) the ring of rational integers (resp. p-adic integers), and also denote by $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) the rational number field (resp. the p-adic number field). We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and fix embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C},$$

where $\mathbb{C}$ denotes the complex number field.

We assume that every ring is commutative with unity. For a ring R and $n \in \mathbb{N}$, we use the following notation:

$$\begin{aligned} \mathrm{M}_n(R) &= \{(n \times n)\text{-matrices with entries in } R\}, \\ \mathrm{GL}_n(R) &= \{M \in \mathrm{M}_n(R) | M \text{ is an invertible matrix}\}, \\ \mathrm{SL}_n(R) &= \{M \in \mathrm{GL}_n(R) | \det(M) = 1\}. \end{aligned}$$

Moreover, if R is a subring of $\mathbb{R}$, we put

$$\mathrm{GL}_n^+(R) = \{M \in \mathrm{GL}_n(R) | \det(M) > 0\}.$$

Let $\mathfrak{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ be the upper half plane and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ the extended upper half plane obtained by adding the cusps. Then $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathfrak{H}$ by

$$\alpha z = \frac{az+b}{cz+d}$$

for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathfrak{H}$. Let $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

The principal congruence subgroups are the subgroup $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma(N) = \left\{ \alpha \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$

where N is a positive integer. A congruence subgroup is a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ containing a principal congruence group. The smallest integer $N > 0$ for which

$$\Gamma(N) \subset \Gamma$$

is called the level of Γ.

We will be mostly interested in the following special congruence subgroups:

$$\Gamma_0(N) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\},$$
$$\Gamma_1(N) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv d \equiv 1 \bmod N \right\}.$$

Let k be a positive integer ≥ 2. For any function f on $\mathfrak{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, we define the function $f|_k\gamma$ on $\mathfrak{H}$ by

$$f|_k\gamma(z) = \det(\gamma)^{k-1} f(\gamma z)(cz+d)^{-k}.$$

We simply write $f|_k\gamma$ for $f|\gamma$ if there is no risk of confusion. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and N a positive integer such that $\Gamma(N) \subset \Gamma$. Any holomorphic function on $\mathfrak{H}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma(N)$ has the Fourier expansion of the form:

$$\sum_{n=0}^{\infty} a(n,f)\, e\left(\frac{nz}{N}\right),$$

where $e(z) = \exp(2\pi\sqrt{-1}z)$.

We define the space $M_k(\Gamma, \mathbb{C})$ of modular forms of weight k with respect to Γ to be the space of holomorphic functions f on $\mathfrak{H}$ satisfying the following conditions:

(a) $f|_k\gamma = f$ for all $\gamma \in \Gamma$.