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THE CONTRIBUTIONS OF HILBERT AND DEHN TO NON-ARCHIMEDEAN GEOMETRIES AND THEIR IMPACT ON THE ITALIAN SCHOOL

Cinzia Cerroni

Abstract. — In this paper we investigate the contribution of Dehn to the development of non-Archimedean geometries. We will see that it is possible to construct some models of non-Archimedean geometries in order to prove the independence of the continuity axiom and we will study the interrelations between Archimedes' axiom and Legendre's theorems. Some of these interrelations were also studied by Bonola, who was one of the very few Italian scholars to appreciate Dehn's work. We will see that, if Archimedes' axiom does not hold, the hypothesis on the existence and the number of parallel lines through a point is not related to the hypothesis on the sum of the inner angles of a triangle. Hilbert himself returned to this problem giving a very interesting model of a non-Archimedean geometry in which there are infinitely many lines parallel to a fixed line through a point while the sum of the inner angles of a triangle is equal to two right angles.

Résumé (Les contributions de Hilbert et de Dehn aux géométries non-archimédiennes et leur impact sur l'école italienne)

Cet article présente les contributions de Max Dehn au développement des géométries non archimédiennes. Un moyen pour montrer l'indépendance de l'axiome d'Archimède par rapport aux axiomes d'incidence et d'ordre est de construire des modèles de géométries non archimédiennes. Les travaux de Max Dehn dans ce champ concernent pour l'essentiel les relations entre l'axiome d'Archimède et les théorèmes de Legendre. Quelques-unes de ces liaisons ont

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Mots clefs. — David Hilbert, Max Dehn, Federico Enriques, Roberto Bonola, géométrie non-archimédienne.

été aussi étudiées par Bonola, un étudiant d'Enriques, qui est parmi les rares Italiens à avoir apprécié le travail de Dehn. Un des principaux résultats, lorsque l'axiome d'Archimède n'est pas satisfait, est que l'axiome des parallèles est indépendant de celui de la somme des angles internes d'un triangle. Hilbert lui-même revint sur ce problème en construisant un modèle de géométrie non archimédienne dans lequel il y a une infinité de droites passant par un point et parallèles à une droite donnée, alors que la somme des angles internes d'un triangle est égale à deux angles droits.

1. INTRODUCTION

The *Grundlagen der Geometrie* (1899) by David Hilbert triggered a new phase in geometrical research. The analysis of the interrelation and independence of the axioms gave rise to new geometries, the study of which acquired the same importance as that of classical, Euclidean geometry. This work is part of a research project on the creation of new geometries in the first half of the 20$^{\text{th}}$ century and on the analysis of their interrelations with algebra, which yielded its first result in a work [Cerroni 2004] on the study of non-Desarguesian geometries. The present paper aims to pursue this line of research, dealing with non-Archimedean geometries.

The starting point of research on non-Archimedean geometries was the investigation of the independence of Archimedes' axiom from other axioms.

As it is well known, Archimedes' axiom states that if A and B are two segments, with A smaller than B $(A < B)$, then there exists a positive integer n such that n times A is greater than B $(nA > B)$.

Giusepppe Veronese made the first attempt to construct a model of non-Archimedean geometry [1]. In *Fondamenti di geometria* [Veronese 1891], he constructed, in abstract manner, a geometry in which he postulated the existence of a segment which is infinitesimal with respect to another, and where the straight line of geometry is not equated with the continuous straight line of Dedekind. It is not our aim to develop in depth the study of Veronese's work, for which we refer to the literature [2] and to a forthcoming paper. So, we limit ourselves to sketch the contributions by

1 For a study of the emergence of non-Archimedean systems of magnitudes see [Ehrlich 2006]

2 For study of Veronese's non-Archimedean straight line see [Busulini 1969/70] and [Cantù 1999].

Veronese and we want to go directly to Dehn's contributions and to its influence on Italian geometers.

Veronese attacked the question about the existence of a segment which is infinitesimal in respect to another and so about the existence of a non-Archimedean geometry while analyzing the independence of Archimedes' axiom from the others:

> "[...] The question about the existence of a segment which is infinitesimal in respect to another is ancient; but neither the supporters neither the opponents have proved the possibility or the impossibility of this idea, because they did not put the question in a right way, complicating it with philosophical considerations unrelated with it. Instead, it has to be put in the same way than those about the parallel axiom and the space dimensions; that is, if all the axioms hold, is Archimedes' axiom a consequence of the others? Or in other words, let A and B be two segments $(A < B)$, does there exist a geometry in which in general it is not true that $An > B$, where n is a positive integer $1, 2, \dots n$?
>
> If one takes Dedekind's axiom as the continuity axiom, or if one maps the points of the line in to the real numbers, then the previous relation is a consequence of it. But I gave a new definition of the continuity axiom that does not contain Archimedes' axiom [...]" [3] [Veronese 1898, p. 79].

Therefore, the central problem for the author is to find a definition of the continuity axiom that does not contain Archimedes' axiom [4], that is to define a non-Archimedean continuous [5]. The geometric continuous is the way to define an abstract continuous independent from the numeric continuous. Veronese supposed, in a system of axioms and in his definition of continuity, the existence of infinite limited segments and so the existence

3 "[...] *La questione del segmento infinitesimo attuale è antica; ma né i sostenitori né gli oppositori di tale idea ne hanno mai provata la possibilità o la impossibilità geometrica, perché essi non hanno posta la questione in modo chiaro e determinato, complicandola talvolta con considerazioni filosofiche ad essa estranee. Essa invece va posta in modo analogo a quelle relative ai postulati delle parallele e delle dimensioni dello spazio; vale a dire dati tutti i postulati necessari per costruire la figura corrispondente al campo della nostra osservazione esterna e considerati come possibili tutti quei postulati che non contraddicono ai precedenti e non si contraddicono fra loro, il postulato di Archimede è esso conseguenza degli altri? O in altre parole è possibile una geometria nella quale dati due segmenti A e B $(A < B)$ non obbediscano in generale alla relazione $An > B$, essendo n un numero intero qualunque della serie $1, 2, \dots, n, \dots$? Se si dà il postulato della continuità nella forma proposta da Dedekind, o facendo corrispondere biunivocamente i punti della retta ai numeri reali ordinari, allora detta relazione si può considerare come un'immediata conseguenza di esso. Ma io diedi della continuità un'altra forma che pur mantenendo i caratteri del continuo rettilineo non racchiude quella di Archimede* [...]". All the translations are by the author of the paper.

4 The first work of Veronese on this topic is [Veronese 1890].

5 For a study of Veronese's non-Archimedean continuous see [Cantù 1999].