

quatrième série - tome 43 fascicule 6 novembre-décembre 2010

*ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE*

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RABINOWITZ FLOER HOMOLOGY AND SYMPLECTIC HOMOLOGY

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ABSTRACT. – The first two authors have recently defined Rabinowitz Floer homology groups $RFH_*(M, W)$ associated to a separating exact embedding of a contact manifold (M, ξ) into a symplectic manifold (W, ω) . These depend only on the bounded component V of $W \setminus M$. We construct a long exact sequence in which symplectic cohomology of V maps to symplectic homology of V , which in turn maps to Rabinowitz Floer homology $RFH_*(M, W)$, which then maps to symplectic cohomology of V . We compute $RFH_*(ST^*L, T^*L)$, where ST^*L is the unit cosphere bundle of a closed manifold L . As an application, we prove that the image of a separating exact contact embedding of ST^*L cannot be displaced away from itself by a Hamiltonian isotopy, provided $\dim L \geq 4$ and the embedding induces an injection on π_1 .

RÉSUMÉ. – Étant donné un plongement exact et séparant d’une variété de contact (M, ξ) dans une variété symplectique (W, ω) , les deux premiers auteurs ont défini des groupes d’homologie dits de Rabinowitz Floer $RFH_*(M, W)$. Ceux-ci dépendent uniquement de la composante bornée V de $W \setminus M$. Nous construisons une suite exacte longue dans laquelle la cohomologie symplectique de V est envoyée vers l’homologie symplectique de V , qui à son tour est envoyée vers l’homologie de Rabinowitz Floer $RFH_*(M, W)$, qui finalement est envoyée vers la cohomologie symplectique de V . Nous calculons $RFH_*(ST^*L, T^*L)$ pour le fibré cotangent unitaire ST^*L d’une variété compacte sans bord L . Nous démontrons que l’image d’un plongement exact et séparant de ST^*L ne peut pas être disjointe d’elle-même par une isotopie hamiltonienne, à condition que le plongement induise une injection sur le groupe fondamental et $\dim L \geq 4$.

1. Introduction

Let (W, λ) be a complete convex exact symplectic manifold, with symplectic form $\omega = d\lambda$ (see Section 3 for the precise definition). An embedding $\iota : M \hookrightarrow W$ of a contact manifold (M, ξ) is called an *exact contact embedding* if there exists a 1-form α on M such that $\ker \alpha = \xi$

K. Cieliebak is partially supported by DFG grant CI 45/1-3. U. Frauenfelder is partially supported by New Faculty Research Fund of Seoul National University. A. Oancea is partially supported by ANR project “Floer Power” ANR-08-BLAN-0291-03.

and $\alpha - \lambda|_M$ is exact. We identify M with its image $\iota(M)$. We assume throughout the paper that $W \setminus M$ consists of two connected components and denote the bounded component of $W \setminus M$ by V . One can classically [29] associate to such an exact contact embedding the symplectic (co)homology groups $SH_*(V)$ and $SH^*(V)$. We refer to Section 2 for the definition and basic properties, and to [25] for a recent survey.

The first two authors have recently defined for such an exact contact embedding Floer homology groups $RFH_*(M, W)$ for the Rabinowitz action functional [10]. We refer to Section 3 for a recap of the definition and of some useful properties. We will show in particular that these groups do not depend on W , but only on V (the same holds for $SH_*(V)$ and $SH^*(V)$). We shall use in this paper the notation $RFH_*(V)$ and call them *Rabinowitz Floer homology* groups.

REMARK 1.1. – All (co)homology groups are taken with field coefficients. Without any further hypotheses on the first Chern class $c_1(V)$ of the tangent bundle, the symplectic (co)homology and Rabinowitz Floer homology groups are $\mathbb{Z}_2$ -graded. If $c_1(V) = 0$ they are $\mathbb{Z}$ -graded, and if $c_1(V)$ vanishes on $\pi_2(V)$ the part constructed from contractible loops is $\mathbb{Z}$ -graded. This $\mathbb{Z}$ -grading on Rabinowitz Floer homology differs from the one in [10] (which takes values in $\frac{1}{2} + \mathbb{Z}$) by a shift of $1/2$ (see Remark 3.2).

Our purpose is to relate these two constructions. The relevant object is a new version of symplectic homology, denoted by $\check{S}H_*(V)$, associated to “ $\check{V}$ -shaped” Hamiltonians like the one in Figure 1 on page 976 below. This version of symplectic homology is related to the usual ones via the long exact sequence in the next theorem.

THEOREM 1.2. – *There is a long exact sequence*

$$(1) \quad \cdots \longrightarrow SH^{-*}(V) \longrightarrow SH_*(V) \longrightarrow \check{S}H_*(V) \longrightarrow SH^{-*+1}(V) \longrightarrow \cdots$$

One can think of the long exact sequence (1) in two complementary ways: either as measuring the defect from being an isomorphism for the canonical map $SH^{-*}(V) \rightarrow SH_*(V)$, defined in Section 2.7, or, in view of Theorem 1.5 below, as being an obstruction against having an isomorphism $RFH_*(V) \simeq SH_*(V) \oplus SH^{-*+1}(V)$. An interesting fact is that we have a very precise description of the map $SH^{-*}(V) \rightarrow SH_*(V)$. To state it, let us recall that there are canonical morphisms induced by truncation of the range of the action $H_{*+n}(V, \partial V) \xrightarrow{c_*} SH_*(V)$ and $SH^*(V) \xrightarrow{c^*} H^{*+n}(V, \partial V)$ (see [29] or Lemma 2.1 below).

PROPOSITION 1.3. – *The map $SH^{-*}(V) \rightarrow SH_*(V)$ fits into a commutative diagram*

$$(2) \quad \begin{array}{ccc} SH^{-*}(V) & \longrightarrow & SH_*(V) \\ c_* \downarrow & & \uparrow c_* \\ H^{-*+n}(V, \partial V) & \longrightarrow & H_{*+n}(V, \partial V) \end{array}$$

in which the bottom arrow is the composition of the map induced by the inclusion $V \hookrightarrow (V, \partial V)$ with the Poincaré duality isomorphism

$$H^{-*+n}(V, \partial V) \xrightarrow{PD} H_{*+n}(V) \xrightarrow{\text{incl}_*} H_{*+n}(V, \partial V).$$

We also define in Section 2.7 truncated versions $\check{S}H_*^{\geq 0}(V)$ and $\check{S}H_*^{\leq 0}(V)$ of the symplectic homology groups $\check{S}H_*(V)$.

PROPOSITION 1.4. – *There are commuting diagrams of long exact sequences as below, where PD denotes Poincaré duality and the top exact sequence is the (co)homological long exact sequence of the pair $(V, M = \partial V)$:*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{*+n}(V) & \longrightarrow & H_{*+n}(V, M) & \longrightarrow & H_{*+n-1}(M) \longrightarrow H_{*+n-1}(V) \longrightarrow \cdots \\ & & \parallel \text{PD} & & \downarrow & & \downarrow & & \parallel \text{PD} \\ \cdots & \longrightarrow & H^{-*+n}(V, M) & \longrightarrow & SH_*(V) & \longrightarrow & \check{S}H_*^{\geq 0}(V) & \longrightarrow & H^{-*+1+n}(V, M) \longrightarrow \cdots \end{array}$$

and

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^{-*+n}(V, M) & \longrightarrow & H^{-*+n}(V) & \longrightarrow & H^{-*+n}(M) \longrightarrow H^{-*+n+1}(V) \longrightarrow \cdots \\ & & \uparrow & & \parallel \text{PD} & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & SH^{-*}(V) & \longrightarrow & H_{*+n}(V, M) & \longrightarrow & \check{S}H_*^{\leq 0}(V) & \longrightarrow & SH^{-*+1}(V) \longrightarrow \cdots \end{array}$$

The main result of this paper is the following.

THEOREM 1.5. – *We have an isomorphism*

$$RFH_*(V) \simeq \check{S}H_*(V).$$

Theorem 1.5 is proved in Section 6. It follows that the Rabinowitz Floer homology groups fit into a long exact sequence

$$(3) \quad \cdots \longrightarrow SH^{-*}(V) \longrightarrow SH_*(V) \longrightarrow RFH_*(V) \longrightarrow SH^{-*+1}(V) \longrightarrow \cdots$$

We also recall the following vanishing result for Rabinowitz Floer homology from [10].

THEOREM 1.6 ([10, Theorem 1.2]). – *If $M = \partial V$ is Hamiltonianly displaceable in W , then*

$$RFH_*(V) = 0.$$

To state the next corollary, we recall that the symplectic (co)homology and Rabinowitz Floer homology groups decompose as direct sums

$$SH_*(V) = \oplus_c SH_*^c(V), \quad SH^*(V) = \oplus_c SH_c^*(V), \quad RFH_*(V) = \oplus_c RFH_*^c(V)$$

indexed over free homotopy classes of loops in V . We denote the free homotopy class of the constant loops by $c = 0$.

COROLLARY 1.7. – *Assume $M = \partial V$ is Hamiltonianly displaceable in W .*

– *For $c \neq 0$ we have*

$$SH_*^c(V) = 0, \quad SH_c^*(V) = 0.$$

– *Suppose that $c_1(W)|_{\pi_2(W)} = 0$. Then for $c = 0$ we have*

$$(4) \quad SH_*^{c=0}(V) = 0, \quad SH_{c=0}^*(V) = 0$$

if $ \geq n$ or $* \leq -n$. Moreover, if V is Stein then (4) holds for $* \neq 0$, and if V is Stein subcritical then (4) holds for all $* \in \mathbb{Z}$.*

Proof. – The long exact sequence (3) splits into a direct sum of long exact sequences, indexed over free homotopy classes of loops in V . The assumption that M is Hamiltonianly displaceable implies $RFH_*(V) = 0$, hence the map $SH_c^{-*}(V) \rightarrow SH_*^c(V)$ is an isomorphism for any c .

We now use the commutative diagram in Proposition 1.3 and the fact that the canonical map $c_* : H_{*+n}(V, \partial V) \rightarrow SH_*(V)$ takes values into the direct summand $SH_*^{c=0}(V)$, and similarly, the map $c^* : SH^*(V) \rightarrow H^{*+n}(V, \partial V)$ factors through $SH_{c=0}^*(V)$ (see Lemma 2.1 and Lemma 2.4 below).

Let us assume $c \neq 0$. Then the above discussion shows that the map $SH_c^{-*}(V) \rightarrow SH_*^c(V)$ is at the same time an isomorphism and vanishes. This implies the conclusion.

Let us now assume $c = 0$ and $c_1(W)|_{\pi_2(W)} = 0$, so that all homology groups are $\mathbb{Z}$ -graded. By Proposition 1.3, the map $SH_{c=0}^{-*}(V) \rightarrow SH_*^{c=0}(V)$ is the composition

$$SH_{c=0}^{-*}(V) \rightarrow H^{-*+n}(V, \partial V) \simeq H_{*+n}(V) \rightarrow H_{*+n}(V, \partial V) \rightarrow SH_*^{c=0}(V),$$

and therefore vanishes if $H_{*+n}(V) = 0$ or $H_{*+n}(V, \partial V) \cong H^{n-*}(V) = 0$. This is always the case if $* \geq n$ or $* \leq -n$. If V is Stein, this holds if $* \neq 0$, and if V is Stein subcritical, this holds for all $* \in \mathbb{Z}$. The conclusion follows. $\square$

COROLLARY 1.8 ([8]). – *If V is Stein subcritical and $c_1(V)|_{\pi_2(V)} = 0$, then $SH_*(V) = 0$.*

Proof. – Any compact set in a subcritical Stein manifold is Hamiltonianly displaceable [3]. Thus V is displaceable in its symplectic completion (see Section 2 for the definition), and therefore $SH_*(V) = 0$ by Corollary 1.7. $\square$

Remark. – The original proof of Corollary 1.8 in [8] uses a handle decomposition for W . The proof given above only uses the fact that the subcritical skeleton can be displaced from itself [3]. On the other hand, the proof given above uses the grading in an essential way and hence only works under the hypothesis $c_1(V)|_{\pi_2(V)} = 0$, whereas the original proof does not need this assumption.

Remark. – Ritter proved in [18, Theorem 96] that Rabinowitz Floer homology vanishes if and only if symplectic homology vanishes, if and only if symplectic cohomology vanishes. This is essentially a consequence of the unital ring structure of symplectic homology, with the unit living in degree $* = n$.

COROLLARY 1.9 (Weinstein conjecture in displaceable manifolds)

Assume that V is Hamiltonianly displaceable in W and $c_1(W)|_{\pi_2(W)} = 0$. Then any hypersurface of contact type $\Sigma \subset V$ carries a closed characteristic, i.e. a closed integral curve of the line distribution $\ker \omega|_{\Sigma}$.

Proof. – This follows from the fact that $SH_{c=0}^n(V) = 0$, as proved in Corollary 1.7 above. In particular the canonical map $SH_{c=0}^n(V) \rightarrow H^{2n}(V, \partial V)$ vanishes, and thus V satisfies the Strong Algebraic Weinstein Conjecture in the sense of Viterbo [29]. The conclusion is then a consequence of the Main Theorem in [29] (see also [16, Theorem 4.10] for details). $\square$