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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

## SECOND ORDER ELLIPTIC OPERATORS WITH COMPLEX BOUNDED MEASURABLE COEFFICIENTS IN $L^p$ , SOBOLEV AND HARDY SPACES

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**ABSTRACT.** – Let  $L$  be a second order divergence form elliptic operator with complex bounded measurable coefficients. The operators arising in connection with  $L$ , such as the heat semigroup and Riesz transform, are not, in general, of Calderón-Zygmund type and exhibit behavior different from their counterparts built upon the Laplacian. The current paper aims at a thorough description of the properties of such operators in  $L^p$ , Sobolev, and some new Hardy spaces naturally associated to  $L$ .

First, we show that the known ranges of boundedness in  $L^p$  for the heat semigroup and Riesz transform of  $L$ , are sharp. In particular, the heat semigroup  $e^{-tL}$  need not be bounded in  $L^p$  if  $p \notin [2n/(n+2), 2n/(n-2)]$ . Then we provide a complete description of *all* Sobolev spaces in which  $L$  admits a bounded functional calculus, in particular, where  $e^{-tL}$  is bounded.

Secondly, we develop a comprehensive theory of Hardy and Lipschitz spaces associated to  $L$ , that serves the range of  $p$  beyond  $[2n/(n+2), 2n/(n-2)]$ . It includes, in particular, characterizations by the sharp maximal function and the Riesz transform (for certain ranges of  $p$ ), as well as the molecular decomposition and duality and interpolation theorems.

**RÉSUMÉ.** – Soit  $L$  un opérateur elliptique du second ordre de formes de divergence, à coefficients complexes bornés et mesurables. Les opérateurs associés à  $L$  tels que le semi-groupe de la chaleur ou la transformée de Riesz ne sont en général pas de type Calderón-Zygmund et présentent des comportements différents de leurs analogues construits à partir du laplacien. Cet article a pour objectif de décrire de manière exhaustive les propriétés de ces opérateurs dans  $L^p$ , dans les espaces de Sobolev ainsi que dans certains nouveaux espaces de Hardy naturellement associés à  $L$ .

Tout d'abord, nous montrons que les plages de valeurs connues pour lesquelles ces opérateurs sont bornés en norme  $L^p$  sont strictes. En particulier, le semi-groupe de la chaleur et la transformée de Riesz ne sont pas obligatoirement bornés si  $p \notin [2n/(n+2), 2n/(n-2)]$ . Nous fournissons ensuite une description complète de *tous* les espaces de Sobolev pour lesquels  $L$  admet un calcul fonctionnel borné, en particulier, pour lesquels  $e^{-tL}$  est borné.

Puis, nous développons une théorie extensive des espaces de Hardy et de Lipschitz associés à  $L$ , pour les valeurs de  $p$  hors de  $[2n/(n+2), 2n/(n-2)]$ . Cette théorie comprend, en particulier, des caractérisations par la fonction maximale « dièse » et par la transformée de Riesz (pour certaines plages de  $p$ ), ainsi que leur décomposition moléculaire, leur dualité et les théorèmes d'interpolation.

## 1. Introduction

Let  $A$  be an  $n \times n$  matrix with entries

$$(1.1) \quad a_{jk} : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad j = 1, \dots, n, \quad k = 1, \dots, n,$$

satisfying the ellipticity condition

$$(1.2) \quad \lambda |\xi|^2 \leq \Re A\xi \cdot \bar{\xi} \quad \text{and} \quad |A\xi \cdot \bar{\zeta}| \leq \Lambda |\xi| |\zeta|, \quad \forall \xi, \zeta \in \mathbb{C}^n,$$

for some constants  $0 < \lambda \leq \Lambda < \infty$ . For such matrices  $A$ , our aim in this paper is to present a detailed investigation of Hardy spaces and their duals associated to the second order divergence form operator

$$(1.3) \quad Lf := -\operatorname{div}(A\nabla f),$$

which we interpret in the usual weak sense via a sesquilinear form.

In the case that  $A$  is the  $n \times n$  identity matrix (i.e., so that  $L$  is the usual Laplacian  $\Delta := -\operatorname{div} \cdot \nabla$ ), this theory reduces to the classical Hardy space theory of Stein-Weiss [55] and Fefferman-Stein [32]. For more general operators  $L$  whose heat kernel satisfies a pointwise Gaussian upper bound, an adapted Hardy space theory has been introduced by Auscher, Duong and McIntosh [9], and by Duong and Yan, [27], [28]. In the absence of such pointwise kernel bounds, the theory has been developed more recently in [11] by Auscher, McIntosh and Russ (when  $L$  is the Hodge-Laplace operator on a manifold with doubling measure), and in [40] by the first two authors of the present paper, for the complex divergence form elliptic operators considered here. In [11, 40], the pointwise Gaussian bounds are replaced by the weaker “Gaffney estimates” (cf. (2.21) and (2.24) below), whose  $L^2$  version is a refined parabolic “Caccioppoli” inequality which may also be proved via integration by parts using only ellipticity and the divergence form structure of  $L$ . The present paper may be viewed in part as a sequel to [40], in which we extend results for the case  $p = 1$  given there, to the case of general  $p$  (although we also obtain here some results, pertaining to the characterization of adapted Hardy spaces via Riesz transforms, that are new even in the case  $p = 1$ ). In particular, it is in the nature of our present setting, in which pointwise kernel bounds may fail, that the Hardy space theory for  $p > 1$  becomes non-trivial (i.e., the  $L$ -adapted  $H^p$  spaces may not coincide with  $L^p$ , even when  $p > 1$ ). We shall return to this point momentarily. We note also that general non-negative self-adjoint operators satisfying an  $L^2$  Gaffney estimate have recently been treated in [38].

We now proceed to discuss some relevant history, and to present a more detailed overview of the paper. In [10], the authors solved a long-standing conjecture, known as the Kato problem, by identifying the domain of the square root of  $L$ . More precisely, they showed that the domain of  $\sqrt{L}$  is the Sobolev space  $W^{1,2}(\mathbb{R}^n) = \{f \in L^2 : \nabla f \in L^2\}$  with

$$(1.4) \quad \|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}.$$

In particular, the Riesz transform  $\nabla L^{-1/2}$  is bounded in  $L^2(\mathbb{R}^n)$ .

Since then, substantial progress has been made in the development of the  $L^p$  theory of elliptic operators of the type described above. Let us define

$$p_-(L) := \inf\{p : \nabla L^{-1/2} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)\}.$$

It is now known that  $1 \leq p_-(L) < 2n/(n+2)$  (with  $1 < p_-(L)$  for some  $L$ ; we shall return to the latter point momentarily), and that there exists  $\varepsilon(L) > 0$  such that

$$(1.5) \quad \nabla L^{-1/2} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \iff p_-(L) < p < 2 + \varepsilon(L),$$

(given (1.4) as a starting point, (1.5) with  $p_-(L) < 2n/(n+2)$  is established by combining the results and methods of [39] or [18] with those of [6]; see also [5], [13], Chapter 4 of [14], and [17] for related theory). Moreover, again given (1.4) as a starting point, one has the reverse inequality

$$(1.6) \quad \|\sqrt{L}f\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad \text{for } (p_-(L))_* < p < (p_-(L^*))',$$

where in general  $p_* := pn/(p+n)$  denotes the “lower” Sobolev exponent, and as usual  $p' := p/(p-1)$  is the exponent dual to  $p$ . The case  $p < 2$  of (1.6) is due to Auscher [6], while the case  $p > 2$  is simply dual to the adjoint version of (1.5). Combining (1.5) and (1.6), we have that

$$(1.7) \quad \|\sqrt{L}f\|_{L^p(\mathbb{R}^n)} \approx \|\nabla f\|_{L^p(\mathbb{R}^n)} \iff p_-(L) < p < 2 + \varepsilon.$$

One of the main goals of the present paper is to understand the sense in which (1.7) extends to the range  $p \leq p_-(L)$ . This extension may be viewed as solving the Kato problem below the critical exponent  $p_-(L)$ . We discuss this question in more detail in Subsection 1.2 below; the proofs are given in Section 5 (cf. Theorem 5.2).

Let us now discuss optimality of the range of  $p$  in (1.5) (hence also that in (1.7)), for the entire class of  $L$  under consideration. Even in the case of real symmetric coefficients, the upper bound cannot be improved, in general: for each  $p > 2$ , Kenig<sup>(1)</sup> has constructed an operator  $L$  whose Riesz transform is not bounded in  $L^p$ . In addition, the counterexamples in [50], [8], [25] showed that for some elliptic operator  $L$  satisfying (1.1)–(1.3) there is a  $p \in (1, 2)$  such that the Riesz transform is not bounded in  $L^p$ ; i.e., for such  $L$ , one has  $p_-(L) > 1$ . Moreover, the latter fact permeates all the  $L^p$  results in the theory: as shown in [6],  $p_-(L)$  is also the lower bound for the respective intervals of  $p$  for which the heat semigroup and the  $L$ -adapted square function (cf. (1.10) below) are  $L^p$  bounded, and for which the semigroup enjoys  $L^p \rightarrow L^2$  off diagonal estimates. However, identification of the sharp lower bound  $p_-(L)$  remained an open problem (posed, along with related questions, in [6], Conjecture 3.14, and in [4], Problem 1.4, Problem 1.5, Problem 1.13).

In Section 2 of the present paper, we observe that the example constructed by Frehse in [34] may be used to resolve these remaining sharpness issues, i.e., to show that  $p_\pm(L) = 2n/(n \mp 2) \pm \varepsilon_\pm(L)$ , where  $(p_-(L), p_+(L))$  is the interior of the interval of  $L^p$  boundedness of the heat semigroup  $e^{-tL}$ ,  $t > 0$ . More precisely, we have

$$(1.8) \quad \forall p \notin [2n/(n+2), 2], \exists L \text{ with } \nabla L^{-1/2} : L^p(\mathbb{R}^n) \not\rightarrow L^p(\mathbb{R}^n),$$

$$(1.9) \quad \forall p \notin [2n/(n+2), 2n/(n-2)], \exists L \text{ with } e^{-tL} : L^p(\mathbb{R}^n) \not\rightarrow L^p(\mathbb{R}^n).$$

It follows, in particular, that in dimensions  $n \geq 3$ , the kernel of the heat semigroup may fail to satisfy the pointwise Gaussian estimate

$$|K_t(x, y)| \leq Ct^{-n/2} e^{-c|x-y|^2/t}, \quad t > 0 \text{ and } x, y \in \mathbb{R}^n.$$

<sup>(1)</sup> Kenig’s Example is described in [14], Section 4.2.2.

This solves an open problem in [14], p. 33.

Thus, in dimensions  $n > 2$ , the Riesz transform may fail to be bounded in  $L^p$  for some  $p \in (1, 2)$ , as may the heat semigroup  $e^{-tL}$ ,  $t > 0$ , as well as the other natural operators associated with such  $L$  (e.g., square function, non-tangential maximal function). Consequently, in the case that the endpoint  $p_-(L) > 1$ , the  $L$ -adapted Riesz transforms, semigroup and square function cannot be bounded from the classical Hardy space  $H^1$  into  $L^1$ , since interpolation with the known  $L^2$  bound would then produce a contradiction with (1.8), (1.9) (or with the analogous statement for the square function). These operators therefore lie beyond the scope of the Calderón-Zygmund theory and exhibit behavior different to their counterparts built upon the Laplacian.

By analogy to the classical theory then, this motivates the introduction of a family of  $L$ -adapted Hardy spaces  $H_L^p$  for all  $0 < p < \infty$ , *not* equal to  $L^p$  in the range  $p \leq p_-(L)$ , on which the  $L$ -adapted semigroup, Riesz transforms and square function are well behaved, and which comprise a complex interpolation scale including  $L^p$  for  $p_-(L) < p < p_+(L)$ . We note that the endpoint  $p_-(L)$  plays a similar role to the exponent  $p = 1$  in the classical theory.

In particular, in Section 5 we give a suitable Hardy space extension of (1.5) to the case  $p \leq p_-(L)$  (the case  $p = 1$  already appeared in [40]), and, in one of the main results of this paper, we present an appropriate converse, thus obtaining a Riesz transform characterization of  $L$ -adapted  $H^p$  spaces, for some range of  $p$  depending on  $n$ . As observed above, this characterization may be viewed as a sharp extension of the Kato square root estimate (1.4), and of its  $L^p$  version (1.7), to the endpoint  $p_-(L)$  and below. In order to make these notions precise, we should first define our adapted  $H_L^p$  spaces.

### 1.1. Definition of $H_L^p$

The first step in the development of an  $L$ -adapted Hardy space theory, in the case that pointwise kernel bounds may fail<sup>(2)</sup>, was taken in [40] (and independently in [11]), in which the authors considered the model case of  $H_L^1(\mathbb{R}^n)$  and, on the dual side, the appropriate analogue of the space BMO. The definition of  $H_L^1$  given in [40]<sup>(3)</sup> (by means of an  $L$ -adapted square function) can be extended immediately to  $0 < p \leq 2$  and with some additional care to  $2 \leq p < \infty$  as well. To this end, consider the square function associated with the heat semigroup generated by  $L$

$$(1.10) \quad Sf(x) = \left( \iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where, as usual,  $\Gamma(x) = \{(y, t) \in \mathbb{R}^n \times (0, \infty) : |x - y| < t\}$  is a non-tangential cone with vertex at  $x \in \mathbb{R}^n$ . Analogously to [40], we define the space  $H_L^p(\mathbb{R}^n)$  for  $0 < p \leq 2$  as the completion of  $\{f \in L^2(\mathbb{R}^n) : Sf \in L^p(\mathbb{R}^n)\}$  in the norm

$$(1.11) \quad \|f\|_{H_L^p(\mathbb{R}^n)} := \|Sf\|_{L^p(\mathbb{R}^n)}.$$

<sup>(2)</sup> In the presence of *pointwise* Gaussian heat kernel bounds, an  $L$ -adapted  $H^1$  and BMO theory was previously introduced by Duong and Yan [27], [28].

<sup>(3)</sup> And in [11] for  $H_L^p$ ,  $p \geq 1$ .