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MULTIFRACTAL ANALYSIS OF THE DIVERGENCE OF FOURIER SERIES

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ABSTRACT. – A famous theorem of Carleson says that, given any function $f \in L^p(\mathbb{T})$, $p \in (1, +\infty)$, its Fourier series $(S_n f(x))$ converges for almost every $x \in \mathbb{T}$. Beside this property, the series may diverge at some point, without exceeding $O(n^{1/p})$. We define the divergence index at x as the infimum of the positive real numbers β such that $S_n f(x) = O(n^\beta)$ and we are interested in the size of the exceptional sets E_β, namely the sets of $x \in \mathbb{T}$ with divergence index equal to β. We show that quasi-all functions in $L^p(\mathbb{T})$ have a multifractal behavior with respect to this definition. Precisely, for quasi-all functions in $L^p(\mathbb{T})$, for all $\beta \in [0, 1/p]$, E_β has Hausdorff dimension equal to $1 - \beta p$. We also investigate the same problem in $\mathscr{C}(\mathbb{T})$, replacing polynomial divergence by logarithmic divergence. In this context, the results that we get on the size of the exceptional sets are rather surprising.

RÉSUMÉ. – Un célèbre théorème de Carleson nous dit que si une fonction f est de puissance p-ième intégrable ($p > 1$), sa série de Fourier converge presque partout. D'un autre côté, il peut y avoir des points de divergence. Pour un tel point donné x, on peut introduire l'indice de divergence comme étant le plus petit exposant β tel que $S_n f(x) = O(n^\beta)$. On sait que cet indice est au plus égal à $1/p$ et on s'intéresse à la dimension des ensembles exceptionnels de points E_β d'indice de divergence donné β. Nous montrons que quasi-toute fonction de L^p (au sens de Baire) a un comportement multifractal. De façon précise, quasi-sûrement dans L^p, pour tout β, la dimension de Hausdorff de E_β vaut $1-\beta p$. Nous nous intéressons aussi aux fonctions continues pour lesquelles la croissance de $S_n f(x)$ est contrôlée par le logarithme de n. Là encore un indice de divergence (logarithmique) peut être introduit et nous obtenons des résultats surprenants sur la taille des ensembles exceptionnels.

1. Introduction

1.1. Description of the results

The famous theorem of Carleson and Hunt asserts that, when f belongs to $L^p(\mathbb{T})$, $1 < p < +\infty$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the sequence of the partial sums of its Fourier series $(S_n f(x))_{n\geq 0}$ converges for almost every $x \in \mathbb{T}$. On the other hand, it can diverge at some point. This divergence cannot be too fast since, for any $f \in L^p(\mathbb{T})$ and any $x \in \mathbb{T}$, $|S_n f(x)| \leq C_p n^{1/p} \|f\|_p$ (see [14] for instance). In view of these results, a natural question arises. How big can the sets F be such that $|S_n f(x)|$ grows as fast as possible for every $x \in F$? More generally, can we say something on the size of the sets such that $|S_n f(x)|$ behaves like (or as bad as) n^β for some $\beta \in (0, 1/p]$?

To measure the size of subsets of $\mathbb{T}$, we shall use the Hausdorff dimension. Let us recall the relevant definitions (we refer to [5] and to [11] for more on this subject). If $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function satisfying $\phi(0) = 0$ (ϕ is called a *dimension function* or a *gauge function*), the ϕ-Hausdorff outer measure of a set $E \subset \mathbb{R}^d$ is

$$\mathcal{H}^\phi(E) = \lim_{\varepsilon \to 0} \inf_{r \in R_\varepsilon(E)} \sum_{B \in r} \phi(|B|),$$

$R_\varepsilon(E)$ being the set of countable coverings of E with balls B of diameter $|B| \le \varepsilon$. When $\phi_s(x) = x^s$, we write for short $\mathcal{H}^s$ instead of $\mathcal{H}^{\phi_s}$. The Hausdorff dimension of a set E is

$$\dim_{\mathcal{H}}(E) := \sup\{s > 0; \mathcal{H}^s(E) > 0\} = \inf\{s > 0;\ \mathcal{H}^s(E) = 0\}.$$

There exist old results measuring the size of sets of points of divergence of Fourier series. For example, in the book of Kahane and Salem ([9]), we can find such results for functions belonging to a Sobolev space “close“ to L^2. The relevant result in our context is due to J-M. Aubry [1].

Theorem 1.1. – *Let $f \in L^p(\mathbb{T})$, $1 < p < +\infty$. For $\beta \ge 0$, define*

$$\mathcal{E}(\beta, f) = \left\{ x \in \mathbb{T};\ \limsup_{n \to +\infty} n^{-\beta} |S_n f(x)| > 0 \right\}.$$

Then $\dim_{\mathcal{H}}\left(\mathcal{E}(\beta, f)\right) \le 1 - \beta p$. Conversely, given a set E such that $\dim_{\mathcal{H}}(E) < 1 - \beta p$, there exists a function $f \in L^p(\mathbb{T})$ such that, for any $x \in E$, $\limsup_{n \to +\infty} n^{-\beta} |S_n f(x)| = +\infty$.

This result motivated us to introduce the notion of divergence index. For a given function $f \in L^p(\mathbb{T})$ and a given point $x_0 \in \mathbb{T}$, we can define the real number $\beta(x_0)$ as the infimum of the non negative real numbers β such that $|S_n f(x_0)| = O(n^\beta)$. The real number $\beta(x_0)$ will be called the *divergence index* of the Fourier series of f at point x_0. Of course, for any function $f \in L^p(\mathbb{T})$ $(1 < p < +\infty)$ and any point $x_0 \in \mathbb{T}$, $0 \le \beta(x_0) \le 1/p$. Moreover, Carleson's theorem implies that $\beta(x_0) = 0$ almost everywhere and we would like to have precise estimates on the size of the level sets of the function β. These are defined as

$$\begin{aligned} E(\beta, f) &= \{x \in \mathbb{T};\ \beta(x) = \beta\} \\ &= \left\{ x \in \mathbb{T};\ \limsup_{n \to +\infty} \frac{\log |S_n f(x)|}{\log n} = \beta \right\}. \end{aligned}$$

We can ask for which values of β the sets $E(\beta, f)$ are non-empty. This set of values will be called the domain of definition of the divergence spectrum of f. If β belongs to the domain of definition of the divergence spectrum, it is also interesting to estimate the Hausdorff dimension of the sets $E(\beta, f)$. The function $\beta \mapsto \dim_{\mathcal{H}}(E(\beta, f))$ will be called the divergence spectrum of the function f (in terms of its Fourier series). By Aubry's result, $\dim_{\mathcal{H}}(E(\beta, f)) \le 1 - \beta p$ and, for any fixed $\beta_0 \in [0, 1/p)$, for any $\varepsilon > 0$, one can find $f \in L^p(\mathbb{T})$ such that $\dim_{\mathcal{H}}\left(\bigcup_{\beta_0 \le \beta \le 1/p} E(\beta, f)\right) \ge 1 - \beta_0 p - \varepsilon$. Our first main result is that a *typical* function $f \in L^p(\mathbb{T})$ satisfies $\dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p$ for *any* $\beta \in [0, 1/p]$. In particular, f has a multifractal behavior with respect to the summation of its Fourier series, meaning that the domain of definition of its divergence spectrum contains an interval with non-empty interior.

THEOREM 1.2. – *Let* $1 < p < +\infty$.

(1) *For all functions* $f \in L^p(\mathbb{T})$, *for any* $\beta \in [0, 1/p]$, $\dim_{\mathcal{H}} \big(E(\beta, f)\big) \leq 1 - \beta p$.
(2) *For quasi-all functions* $f \in L^p(\mathbb{T})$, *for any* $\beta \in [0, 1/p]$, $\dim_{\mathcal{H}} \big(E(\beta, f)\big) = 1 - \beta p$.

The terminology "quasi-all" used here is relative to the Baire category theorem. It means that this property is true for a residual set of functions in $L^p(\mathbb{T})$. Theorem 1.2 can be compared with other types of results in multifractal analysis, for example regarding Hölder's regularity (see [7]) or fast points for the Brownian motion (see [12]).

In a second part of the paper, we turn to the case of $\mathcal{C}(\mathbb{T})$, the set of continuous functions on $\mathbb{T}$. In that space, the divergence of Fourier series is controlled by a logarithmic factor. More precisely, if (D_n) is the sequence of the Dirichlet kernels, we know that $\|S_n f\|_\infty \leq \|D_n\|_1 \|f\|_\infty$, so that there exists some absolute constant $C > 0$ such that $\|S_n f\|_\infty \leq C\|f\|_\infty \log n$ for any $f \in \mathcal{C}(\mathbb{T})$ and any $n > 1$. As before, one can discuss the size of the sets such that $|S_n f(x)|$ behaves badly, namely like $(\log n)^\beta$, $\beta \in [0, 1]$. More precisely, mimicking the case of the L^p spaces, we introduce, for any $\beta \in [0, 1]$ and any $f \in \mathcal{C}(\mathbb{T})$, the following sets:

$$\mathcal{F}(\beta, f) = \left\{x \in \mathbb{T};\ \limsup_{n\to+\infty} (\log n)^{-\beta} |S_n f(x)| > 0\right\}.$$

When we try to estimate the size of the sets $\mathcal{F}(\beta, f)$, we observe that the Hausdorff dimension is not sufficiently precise. We need a new family of gauge functions. For $s > 0$ and $t \in (0, 1]$, we consider

$$\phi_{s,t}(x) = x^s \exp\left[(\log 1/x)^{1-t}\right].$$

It is not difficult to check that $\phi_{s,t}(x) \leq \phi_{s',t'}(x)$ for small values of x iff

$$s > s' \text{ or } (s = s' \text{ and } t \geq t').$$

The analogue of Aubry's theorem in this context is

PROPOSITION 1.3. – *Let* $\beta \in (0, 1)$ *and* $f \in \mathcal{C}(\mathbb{T})$. *Then, for any* $\gamma > 1 - \beta$,

$$\mathcal{H}^{\phi_{1,\gamma}}\big(\mathcal{F}(\beta, f)\big) = 0.$$

Following the L^p case, we define for $f \in \mathcal{C}(\mathbb{T})$ and $\beta \in [0, 1]$ the level set

$$F(\beta, f) = \left\{x \in \mathbb{T};\ \limsup_{n\to+\infty} \frac{\log |S_n f(x)|}{\log\log n} = \beta\right\}.$$

As indicated in Proposition 1.3 the size of the level sets are measured using the following refinement of the Hausdorff dimension.

DEFINITION 1.4. – Let $E \subset \mathbb{R}^d$. We say that E has *precised Hausdorff dimension* (α, β) if α is the Hausdorff dimension of E and

– $\beta = 0$ if $\mathcal{H}^{\phi_{\alpha,t}}(E) = 0$ for every $t \in (0, 1)$;
– $\beta = \sup\left\{t \in (0, 1);\ \mathcal{H}^{\phi_{\alpha,t}}(E) > 0\right\}$ otherwise.

The precised Hausdorff dimension is a tool to classify sets that have the same Hausdorff dimension. The natural order for the precised dimension (s, t) is the lexicographical order which will be denoted by $\prec$. With respect to this order, we can say that the greater is the set, the greater is the precised dimension. Moreover, if $(s, t) \prec (s', t')$ and $(s, t) \neq (s', t')$, then $\phi_{s',t'} \ll \phi_{s,t}$. It follows that $\mathscr{H}^{\phi_{s',t'}}(E) = 0$ as soon as $\mathscr{H}^{\phi_{s,t}}(E) < \infty$.

Our main theorem on $\mathscr{C}(\mathbb{T})$ is the following:

THEOREM 1.5. – *The following statements are true.*

(1) *For all functions $f \in \mathscr{C}(\mathbb{T})$, for any $\beta \in [0, 1]$, the precised Hausdorff dimension of $F(\beta, f)$ is at most $(1, 1 - \beta)$.*

(2) *For quasi-all functions $f \in \mathscr{C}(\mathbb{T})$, for any $\beta \in [0, 1]$, the precised Hausdorff dimension of $F(\beta, f)$ is equal to $(1, 1 - \beta)$. In particular $\dim_{\mathscr{H}}(F(\beta, f)) = 1$.*

The paper is organized as follows. In the remaining part of this section, we introduce tools which will be needed during the rest of the paper. In Section 2, we prove Theorem 1.2 whereas in Section 3, we prove Theorem 1.5.

We conclude this introduction by mentioning that the companion problem of obtaining similar results for genericity in the sense of prevalence is considered in [2].

1.2. A precised version of Fejér's theorem

Working on Fourier series, we will need results on approximation by trigonometric polynomials. Let $k \in \mathbb{Z}$ and $e_k : t \mapsto e^{2\pi i k t}$, so that, for any $g \in L^1(\mathbb{T})$ and any $n \in \mathbb{N}$,

$$S_n g : t \mapsto \sum_{k=-n}^{n} \langle g, e_k \rangle e_k(t).$$

Let $\sigma_n g$ be the n-th Fejér sum of g,

$$\sigma_n g : t \mapsto \frac{1}{n} \sum_{k=0}^{n-1} S_k g(t).$$

$\sigma_n g$ is obtained by taking the convolution of g with the Fejér kernel

$$F_n : t \mapsto \frac{1}{n} \left(\frac{\sin(n\pi t)}{\sin(\pi t)} \right)^2.$$

If g belongs to $\mathscr{C}(\mathbb{T})$, $(\sigma_n g)_{n \geq 1}$ converges uniformly to g. For our purpose, we need to estimate how quick the convergence is. This is the content of the next lemma (part (1) rectifies a mistake in the proof of Lemma 12 in [1] and requires to replace $\|\theta\|_\infty/4$ in Aubry's version by $\|\theta\|_\infty/2$).

LEMMA 1.6. – *Let θ be a Lipschitz function on $\mathbb{T}$, let $n \in \mathbb{N}$ and let $x \in \mathbb{T}$. Suppose that $\|\theta'\|_\infty \leq n$ and that $\theta(x) = 0$. Then the two following inequalities hold:*

$$|\sigma_n \theta(x)| \leq \frac{1}{4} + \frac{1}{2} \|\theta\|_\infty \quad \textit{for any } n \geq 8 \tag{1}$$

$$|\sigma_n \theta(x)| \leq 4 + \frac{1}{4} \|\theta\|_\infty \quad \textit{for any } n \geq 4. \tag{2}$$