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A STRONG HYPERBOLICITY PROPERTY OF LOCALLY SYMMETRIC VARIETIES

BY YOHAN BRUNEBARBE

ABSTRACT. – We show that all subvarieties of a quotient of a bounded symmetric domain by a sufficiently small arithmetic discrete group of automorphisms are of general type. This result corresponds through the Green-Griffiths-Lang's conjecture to a well-known result of Nadel.

RÉSUMÉ. – On montre que les sous-variétés des quotients de domaines symétriques bornés par un groupe arithmétique suffisamment petit sont toutes de type général, ce qui, rapproché d'un résultat célèbre de Nadel, vérifie la conjecture de Green-Griffiths-Lang pour ces variétés.

1. Introduction

1.1. Main result

An arithmetic locally symmetric variety is by definition a complex analytic space which is isomorphic to a quotient of a bounded symmetric domain $\mathscr{D}$ by an arithmetic lattice $\Gamma \subset \mathrm{Aut}(\mathscr{D})$, see Section 5.1 for a reminder. By a theorem of Baily-Borel [4], every arithmetic locally symmetric variety admits a canonical structure of algebraic variety. In this situation, Tai [3] and Mumford [34] have shown that there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that the algebraic variety $\Gamma' \backslash \mathscr{D}$ is of general type. Recall that an irreducible smooth projective complex variety X is said of general type if it has enough pluricanonical forms to make the canonical rational maps $X \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, \omega_X^{\otimes m})^\vee)$ birational onto their images for $m \gg 1$. In that case, every smooth projective complex variety birational to X is of general type too. An irreducible complex algebraic variety X, not necessarily projective or smooth, is then said of general type if any smooth projective complex variety birational to X is of general type.

The following theorem, which is our main result in this paper, strengthens considerably the result of Tai and Mumford.

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THEOREM 1.1 (Main result). – *Let* $\mathscr{D}$ *be a bounded symmetric domain and* $\Gamma \subset \mathrm{Aut}(\mathscr{D})$ *an arithmetic lattice. Then there exists a subgroup* $\Gamma' \subset \Gamma$ *of finite index such that all subvarieties of* $\Gamma' \backslash \mathscr{D}$ *are of general type.*

REMARKS 1.2. – (i) Theorem 1.1 follows from a stronger statement: it follows from Theorem 5.3 that there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that any smooth projective variety birational to a subvariety of $\Gamma' \backslash \mathscr{D}$ has a big cotangent bundle (in the sense that the tautological line bundle on the corresponding projective bundle is big, see Definition 2.6). But a smooth projective variety with a big cotangent bundle is of general type by the work of Campana, Peternell and Păun, cf. Theorem 2.9.

(ii) Note that in general it is necessary to take a finite index subgroup. Already for $\mathscr{D} = \Delta$ the unit disk in $\mathbb{C}$ and $\Gamma(n) := \ker(\mathrm{Sl}(2,\mathbb{Z}) \to \mathrm{Sl}(2,\mathbb{Z}/n\mathbb{Z}))$, it is well-known and easy to check that the level-n modular curve $Y(n) := \Gamma(n)\backslash\Delta$ is of general type exactly when $n > 6$.

(iii) If $\mathscr{D}$ is a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathscr{D})$ an arithmetic lattice, then by the results of Tai and Mumford already mentioned there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that $\Gamma' \backslash \mathscr{D}$ is of general type. Moreover, it follows from the main result of [36] (see Theorem 1.9 below) that Γ' can be chosen so that all subvarieties *of dimension 1* of $\Gamma' \backslash \mathscr{D}$ are of general type. Our proof of Theorem 1.1 gives a new approach to these results that works also for subvarieties of intermediate dimensions.

(iv) When Γ is a cocompact lattice, Theorem 1.1 is a consequence of [12, Theorem 3.1]. In this case, any torsion-free subgroup $\Gamma' \subset \Gamma$ of finite index satisfies the conclusion.

As a direct application of Borel's algebraization theorem [8, Theorem 3.10], one obtains:

COROLLARY 1.3. – *Let* $\mathscr{D}$ *be a bounded symmetric domain and* $\Gamma \subset \mathrm{Aut}(\mathscr{D})$ *an arithmetic lattice. Then there exists a subgroup* $\Gamma' \subset \Gamma$ *of finite index such that any projective complex variety which admits a non-empty Zariski open subset with an immersive holomorphic map to* $\Gamma' \backslash \mathscr{D}$ *is of general type.*

Note that by [12, Theorem 3.1] a projective complex variety endowed with a (*everywhere defined*) generically immersive holomorphic map to a quotient of a bounded symmetric domain by a torsion-free discrete group Γ of automorphisms is of general type. More generally, if we assume that the variety is only quasi-projective, then it has to be of log-general type by [11, Theorem 0.2]. Corollary 1.3 refines greatly this last result when Γ is an *arithmetic lattice*. Let us emphasize here that the strategy leading to the proof of Theorem 1.1 is different from the approach of [12] and [11]. In particular, this strategy can be used to give a new proof of [12, Theorem 3.1] and [11, Theorem 0.2].

Given two positive intergers g and n, let $\mathscr{A}_g(n)$ denote the moduli stack of principally polarized abelian varieties of dimension g with a level-n structure. The corresponding coarse moduli space $A_g(n)$ is an arithmetic locally symmetric variety whose associated bounded symmetric domain $\mathscr{D}$ is the Harish-Chandra's realization (cf. [33, Theorem 1, p. 94]) of the Siegel half-space of rank g. In this special case we prove the more precise statement:

THEOREM 1.4. – *For any* $g \geq 1$ *and any* $n > 12 \cdot g$, *every subvariety of* $A_g(n)$ *is of general type.*

It follows immediately from Theorem 1.4 that a smooth complete complex algebraic variety which admits a non-empty Zariski-open subset parameterizing a family of principally polarized abelian varieties of dimension g with a level-n structure and whose corresponding period map is generically immersive is of general type as soon as $n > 12 \cdot g$ (we will show more precisely that its cotangent bundle is big, cf. Theorem 4.3).

REMARKS 1.5. – (i) There is an important literature about the Kodaira dimension of the $A_g(n)$ and their subvarieties. In particular, it is known that for $g \geq 7$ the coarse moduli space A_g (and a fortiori every $A_g(n)$ for $n \geq 1$) is of general type. This was first proved for g divisible by 24 by Freitag [21] and then improved to $g \geq 9$ by Tai [47], $g \geq 8$ by Freitag [22] and finally $g \geq 7$ by Mumford [35]. On the other hand, A_g is known to be unirational for $g \leq 5$ (due to [19] for $g = 5$, [15] for $g = 4$, classical for $g \leq 3$). Later, Weissauer [52] proved that for $g \geq 13$ every subvariety of the coarse moduli space A_g of codimension $\leq g - 13$ is of general type. A fortiori, the same result holds for all $A_g(n)$ with $g \geq 13$ and $n \geq 1$. However, A_g always contains the codimension $g-1$ subvariety of negative Kodaira dimension $A_1 \times A_{g-1}$ (being uniruled). On the other hand, it follows from [36, 41] that any curve in $A_g(n)$ is of general type when $n > 6 \cdot g$.

(ii) As this paper was being written, Abramovich and Várilly-Alvarado posted a preprint [2] where they prove (cf. Theorem 1.4 in op. cit.) that for any closed subvariety X in $\mathscr{A}_g$, there exists a level n_X (a priori depending on X) such that the irreducible components of the preimage of X in $\mathscr{A}_g(n)$ are of general type for $n > n_X$. Our result gives an explicit n_X which works for all X.

1.2. The geometric Lang conjecture for the Baily-Borel compactification of arithmetic locally symmetric varieties

Given an arithmetic locally symmetric variety $\Gamma\backslash\mathscr{D}$, we denote by $\overline{\Gamma\backslash\mathscr{D}}^*$ its Satake-Baily-Borel compactification. It is a normal projective variety which contains $\Gamma\backslash\mathscr{D}$ as a Zariski-open dense subset, and the boundary $\overline{\Gamma\backslash\mathscr{D}}^* - \Gamma\backslash\mathscr{D}$ is stratified by arithmetic locally symmetric varieties. For example, in the case of $A_g(n)$, the projective variety $\overline{A_g(n)}^*$ admits a natural stratification by locally closed subvarieties, each of which being canonically isomorphic to $A_{g'}(n)$ for some $0 \leq g' \leq g$ [20, Theorem 2.5, p.252]. Therefore, as a direct application of Theorem 1.4, one obtains:

COROLLARY 1.6. – *For any $g \geq 1$ and any $n > 12 \cdot g$, every subvariety of $\overline{A_g(n)}^*$ is of general type.*

Similarly, as an application of Theorem 1.1 and the construction of the Baily-Borel compactification, one can show:

COROLLARY 1.7. – *Let $\mathscr{D}$ be a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathscr{D})$ an arithmetic lattice. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that all subvarieties of $\overline{\Gamma'\backslash\mathscr{D}}^*$ are of general type.*