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HIGHER ORDER CHEEGER INEQUALITIES FOR STEKLOV EIGENVALUES

BY Asma HASSANNEZHAD AND Laurent MICLO

ABSTRACT. – We prove a lower bound for the k-th Steklov eigenvalues in terms of an isoperimetric constant called the k-th Cheeger-Steklov constant in three different situations: finite spaces, measurable spaces, and Riemannian manifolds. These lower bounds can be considered as higher order Cheeger type inequalities for the Steklov eigenvalues. In particular it extends the Cheeger type inequality for the first nonzero Steklov eigenvalue previously studied by Escobar in 1997 and by Jammes in 2015 to higher order Steklov eigenvalues. The technique we develop to get this lower bound is based on considering a family of accelerated Markov operators in the finite and measurable situations and of mass concentration deformations of the Laplace-Beltrami operator in the manifold setting which converges uniformly to the Steklov operator. As an intermediary step in the proof of the higher order Cheeger type inequality, we define the Dirichlet-Steklov connectivity spectrum and show that the Dirichlet connectivity spectra of this family of operators converges to (or is bounded by) the Dirichlet-Steklov spectrum uniformly. Moreover, we obtain bounds for the Steklov eigenvalues in terms of its Dirichlet-Steklov connectivity spectrum which is interesting in its own right and is more robust than the higher order Cheeger type inequalities. The Dirichlet-Steklov spectrum is closely related to the Cheeger-Steklov constants.

RÉSUMÉ. – Pour tout $k \in \mathbb{N}$, une borne inférieure pour la k-ième valeur propre de Steklov en termes d'une constante isopérimétrique, appelée la k-ième constante de Cheeger-Steklov, est obtenue dans trois situations différentes: espaces finis, espaces mesurables et variétés riemanniennes. Ces bornes inférieures peuvent être considérées comme des inégalités de type Cheeger d'ordre supérieur pour les valeurs propres de Steklov. En particulier, elles étendent l'inégalité de type Cheeger pour la première valeur propre non nulle de Steklov étudiée par Escobar en 1997 et par Jammes en 2015. La technique développée pour obtenir ces bornes inférieures utilise une famille d'opérateurs de Markov accélérés dans les situations finies et mesurables et une famille d'opérateurs de Laplace-Beltrami déformés et concentrés près de la frontière. Lors d'une étape intermédiaire de la preuve de l'inégalité de type Cheeger d'ordre supérieur, nous définissons le spectre de connectivité de Dirichlet-Steklov et nous montrons que les spectres de connectivité de Dirichlet de cette famille d'opérateurs convergent uniformément vers (ou sont bornés par) le spectre de Dirichlet-Steklov. De plus, nous obtenons des bornes pour les valeurs propres de Steklov en termes du spectre de connectivité de Dirichlet-Steklov, ce dernier étant intéressant en lui-même. Il est aussi plus robuste que les inégalités de type Cheeger d'ordre supérieur. Le spectre de Dirichlet-Steklov et les constantes de Cheeger-Steklov sont étroitement liés.

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1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension n with smooth boundary, the Steklov eigenvalue problem is

$$
\begin{cases} \Delta f = 0, & \text{in } M, \\ \frac{\partial f}{\partial \nu} = \sigma f, & \text{on } \partial M, \end{cases} \tag{1}
$$

where $\Delta = \operatorname{div} \nabla$ is the Laplace-Beltrami operator on M and ν is the unit outward normal vector along ∂M. Its spectrum consists of a sequence of nonnegative real numbers with accumulation point only at infinity. We denote the sequence of the Steklov eigenvalues by

$$0 = \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq \cdots \nearrow \infty.$$

The Steklov eigenvalues can be also considered as the eigenvalues of the Dirichlet-to-Neumann operator

$$S : C^\infty(\partial M) \ni f \mapsto \frac{\partial F}{\partial \nu} \in C^\infty(\partial M),$$

where F is the harmonic extension of f into the interior of M. The Steklov problem was first introduced by Steklov [32] in 1902 for bounded domains of the plane. Many interesting developments and progress in the study of the Steklov problem have been attained in recent years. We refer the reader to the survey paper [20] and the references therein for recent developments, and to [23] for a historical account. The relationship between the Steklov eigenvalues and geometry of the underlying space, and also its similarity and difference with the Laplace eigenvalues have been a main focus of interest and a source of inspiration, see for example [13, 16, 10, 21, 14, 17, 22].

The focus of this paper is on obtaining lower bounds for the k-th Steklov eigenvalue σ_k in terms of some isoperimetric constants in three different settings. Our results can be viewed as counterparts of the higher order Cheeger inequalities for the Laplace eigenvalues in discrete setting proved by Lee, Oveis Gharan and Trevisan [26], and in manifold setting by the second author [29]. It is also an extension of Escobar's [13, 14] and Jammes' [22] results for σ_2. We first recall previous results known in this direction.

Let $\mathscr{A}$ denote the family of all nonempty open subsets A of M with piecewise smooth boundary. For every $A \in \mathscr{A}$, let $\mu(A)$ denote its Riemannian measure and $\underline{\mu}(\partial A)$ denote the $(n-1)$-dimensional Riemannian measure of ∂A. We define for every $A \in \mathscr{A}$ the isoperimetric ratios

$$
\eta(A) := \frac{\underline{\mu}(\partial_i A)}{\mu(A)} \qquad \eta'(A) := \frac{\underline{\mu}(\partial_i A)}{\underline{\mu}(\bar{A} \cap \partial M)}, \tag{2}
$$

where $\partial_i A := \partial A \cap \operatorname{Int} M$. Here, $\operatorname{Int} M$ denotes the interior of M. Consider the following isoperimetric constants

$$h_2(M) := \inf_A \max\{\eta(A), \eta(M \setminus A)\} \qquad h'_2(M) := \inf_A \max\{\eta'(A), \eta'(M \setminus A)\}.$$

The constant $h_2(M)$ is the well-known Cheeger constant [8]. Motivated by the celebrated result of Cheeger [8], Escobar [13, 14] introduced the isoperimetric constant $h'_2(M)$ and obtained a lower bound for σ_2 in terms of this isoperimetric constant and the first nonzero eigenvalue of a Robin problem. Recently, Jammes [22] obtained a simpler and more explicit lower bound for σ_2 in terms of an isoperimetric $\tilde{h}'_2(M)$ similar to the one introduced by Escobar, and the Cheeger constant $h_2(M)$:

$$\sigma_2(M) \geq \frac{1}{4}\tilde{h}'_2(M)h_2(M), \tag{3}$$

where $\tilde{h}'_2(M) := \inf\{\eta'(A) \,:\, A \in \mathscr{A}, \text{ and } \mu(A) \leq \frac{\mu(M)}{2}\}$. The proof of (3) is simple and only uses the co-area formula. The constants $h'_2(M)$ and $\tilde{h}'_2(M)$ are interesting geometric quantities. It is an intriguing question if similar geometric lower bounds hold for higher order Steklov eigenvalues σ_k. We give an affirmative answer to this question not only in Riemannian setting but also in the setting of finite and measurable spaces.

Let (M, μ) be a measure space and V a proper subset of M, and let L be an operator acting on a functional subspace $\mathscr{H}$ of $\mathbb{L}^2(\mu)$. Throughout the paper we deal with either of three different settings listed below:

(FS) Finite state spaces: M is a finite set, V is a proper subset of cardinality v, L is a reversible irreducible Markov generator and μ is its unique invariant probability measure. Here $\mathscr{H}$ is the space of functions on M denoted by $\mathscr{F}(M)$.

(MS) Measurable state spaces: (M, μ) is a probability measure space with σ−algebra $\mathscr{M}$, and V is a measurable subset of M such that $0 < \mu[V] < 1$. Here, L is a Markov generator of the form $P - I$, where P is a Markov kernel reversible with respect to μ and I is the identity, and $\mathscr{H} = \mathbb{L}^2(\mu)$.

(RM) Riemannian manifolds: M is a compact Riemannian manifold with smooth boundary ∂M, μ is its Riemannian measure, L is the Laplace-Beltrami operator Δ, and $\mathscr{H}$ is the Sobolev space $H^1(\mu)$. Here V is equal to ∂M.

With the help of L, we define an operator S on V and call it the Steklov operator. In setting (RM), the operator S we consider is in fact the Dirichlet-to-Neumann operator discussed above. For the definition of S in (FS) and (MS) settings we refer to Definitions (9) in Section 2, and (28) in Section 3, respectively. We denote the eigenvalues of S by $\sigma_k(M)$ or simply σ_k. Let $\mathscr{A}$ be a family of admissible sets in M:

— in (FS) settings, $\mathscr{A}$ is the set of all nonempty subsets of M;

— in (MS) setting, $\mathscr{A}$ is the set of all non-negligible elements of $\mathscr{M}$, i.e., $A \in \mathscr{M}$ such that $0 < \mu[A] \leq 1$;

— in (RM) setting, $\mathscr{A}$ is the set of all nonempty open domains A in M such that $\partial_e A := \bar{A} \cap \partial M$ and $\partial_i A := \partial A \cap M$ are smooth manifolds of dimension $n-1$ when they are nonempty.

In (FS) and (MS) settings, we introduce the boundary of any $A \in \mathscr{A}$ via

$$\partial A := \{(x, y) \,:\, x \in A,\ y \in A^c\}$$

and define the following isoperimetric ratios

$$\eta(A) := \frac{\mu(\partial A)}{\mu(A)} \qquad\qquad \eta'(A) := \frac{\mu(\partial A)}{\mu(A \cap V)},$$