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FORMALITY OF THE FUNCTION SPACE OF FREE MAPS INTO AN ELLIPTIC SPACE

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ABSTRACT. — Let X be an n-connected elliptic space and Y a non rationally contractible, finite-type, q-dimensional CW complex, where $q \leq n$. We show that the function space X^Y of free maps from Y into X is formal if and only if the rational cohomology algebra $H^*(X;Q)$ is free, that is, X has the rational homotopy type of a product of odd dimensional spheres.

RÉSUMÉ. — FORMALITÉ DES ESPACES DE FONCTIONS LIBRES DANS UN ESPACE ELLIPTIQUE. — Soient X un espace elliptique n-connexe et Y un CW complexe non rationnellement contractile, de type fini et de dimension $q \leq n$. Nous montrons que l'espace X^Y des fonctions libres de Y dans X est formel si et seulement si l'algèbre $H^*(X,Q)$ est libre, *i.e.* X a le type d'homotopie rationnelle d'un produit de sphères de dimensions impaires.

1. Introduction

D. Sullivan's minimal model $(\wedge V, d)$ satisfies a nilpotence condition on d, *i.e.*, there is a well ordered basis $\{v_i\}_{i\in I}$ of V such that, $i < j$ if $\deg v_i < \deg v_j$ for each $i, j \in I$ and $d(v_i) \in \wedge V_{<i}$. Here $V_{<i}$ denotes the subspace of V generated by basis elements $\{v_j \,;\, j \in I,\ j < i\}$. According to [9, Def. 1.2], $(\wedge V, d)$ is called *normal* if $\operatorname{Ker}[d_{|V}] = \operatorname{Ker}(d_{|V})$ where

$$\operatorname{Ker}\bigl[d_{|V}\bigr] := \bigl\{v_i \in V \,;\, i \in I,\ d(v_i) \text{ is cohomologus to zero in } (\wedge V_{<i}, d)\bigr\}.$$

Let F, E and B be connected nilpotent spaces and let $\mathcal{M}(B)$ be a normal minimal model. In this paper, we say that a rational fibration [7, p. 200]

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$F \xrightarrow{i} E \xrightarrow{\pi} B$ is *M.N* if there is a KS-extension:

$$
\begin{tikzcd}
\mathcal{M}(B) \arrow[r, "\text{inclusion}"] \arrow[d, "\simeq"] & (\mathcal{M}(B) \otimes \wedge V, D) \arrow[r, "\text{projection}"] \arrow[d, "\simeq"] & (\wedge V, \bar{D}) \arrow[d, "\simeq"] \\
A^*(B) \arrow[r, "\pi^*"] & A^*(E) \arrow[r, "i^*"] & A^*(F)
\end{tikzcd}
\tag{1.1}
$$

in which $(\mathcal{M}(B) \otimes \wedge V, D)$ is minimal (*i.e.*, D is decomposable) and normal by a suitable change of KS-basis. Here $A^*(X)$ denotes the rational de-Rham complex of a space X, $\mathcal{M}(F) \cong (\wedge V, \bar{D})$ and "$\simeq$" means quasi-isomorphic, *i.e.*, the map induces an isomorphism in cohomology. We remark that "M.N" is a characteristic of the rational fibration but not of the total space.

Many rational fibrations are M.N. For example, the rational fibration given by a KS-extension:

$$(\wedge(x,y),0) \longrightarrow (\wedge(x,y,z),D) \longrightarrow (\wedge z,0)$$

with $|x| = 3$ (where $|v|$ means $\deg(v)$ for $v \in V$), $|y| = 3$, $|z| = 5$ and $D(z) = xy$ is M.N. Of course, any rationally trivial fibration is M.N. On the other hand, many rational fibrations are not M.N. For example, in the KS-model of the Hopf fibration $S^3 \to S^7 \to S^4$, the model of the total space $(\mathcal{M}(S^4) \otimes \wedge(x_3), D)$ with $|x_3| = 3$ is not even minimal. The rational fibration given by a KS-extension:

$$(\wedge(x,y),d) \longrightarrow (\wedge(x,y,z),D) \longrightarrow (\wedge z,0)$$

with $|x| = 2$, $|y| = 5$, $|z| = 3$, $D(x) = d(x) = 0$, $D(y) = d(y) = x^3$ and $D(z) = x^2$ is minimal but can not be normal by any change of KS-basis.

In the following, a fibration means a rational fibration. A nilpotent space X or the minimal model $\mathcal{M}(X)$ is called (rationally) *formal* if there is a quasi-isomorphism from $\mathcal{M}(X)$ to $(H^*(X;Q),0)$ (see [3]). The reason we consider M.N-type fibrations is that we can then state a necessary (but perhaps not sufficient) condition for the formality of the total space as in [3, Thm 4.1] when the base space is formal (see Lemma 2.3).

A fibration $F \to E \to B$ is called:

- *σ·F* if it has a rational section;
- *W.H.T* if $\pi_*(E) \otimes Q = (\pi_*(B) \otimes Q) \oplus (\pi_*(F) \otimes Q)$ for the rational number field Q and
- *H.T* if it is rationally trivial (see [11]).

The following lemma expresses the relations among these different types of fibrations.

LEMMA 1.1.

1) *"M.N" is embedded in the sequence of implications:*

$$\sigma \cdot F \Longrightarrow M.N \Longrightarrow W.H.T,$$

where the reversed implications are false in general.

2) *If a fibration* $F \to E \to B$ *is* $\sigma{\cdot}F$ *and* E *is formal, then* B *is formal* (*compare* [4, Lemme 2])

Our object of interest is the function space X^Y of free, continuous maps from a connected space Y into a connected space X, endowed with the compact-open topology. Observe that X^Y is infinite dimensional and is connected if X is n-connected and Y is a q-dimensional CW-complex, where $q \leq n$. Furthermore, X^Y is the total space of the fibration:

$$(*) \qquad (X,*)^{(Y,*)} \longrightarrow X^Y \xrightarrow{\ \pi\ } X,$$

where $(X,*)^{(Y,*)}$ is the function space of pointed maps, and π is the evaluation at the base point. We know that (*) has a section s, where $s(x)$ is the constant map at x. Therefore (*) is σ·F. When $Y = S^1$, N. Dupont and M. Vigué-Poirrier proved the following formality result.

THEOREM (see [4, Théorème]). — *Let* X *be a simply connected space where* $H^*(X;Q)$ *is finitely generated. Then* X^{S^1} *is formal iff* $H^*(X;Q)$ *is free, i.e.,* X *has the rational homotopy type of a product of Eilenberg Maclane spaces.*

Our goal in this article is to generalize the theorem of Dupont and Vigué-Poirrier to X^Y, when Y is of finite-type, *i.e.*, $\pi_i(Y) \otimes Q$ is finite-dimensional for all i, provided that X is elliptic, *i.e.*, the total dimensions of $H^*(X;Q)$ and $\pi_*(X) \otimes Q$ are finite. More precisely, we prove the following theorem.

THEOREM 1.2. — *Let* X *be an* n*-connected elliptic space, and let* Y *be a non rationally contractible, finite-type,* q*-dimensional CW complex, where* $q \leq n$*. Then* X^Y *is formal iff* $H^*(X;Q)$ *is free, i.e.,* X *has the rational homotopy type of a product of odd dimensional spheres.*

In proving Theorem 1.2, we use a model due to Brown and Szczarba [2] for the connected component in X^Y of a map $f{:}Y \to X$, which is constructed from minimal models of X, Y and f. We remark that, under the hypotheses of Theorem 1.2, this *non-formalizing tendency* of X^Y does not depend on the rational homotopy type of Y. We cannot easily relax the connectivity hypothesis.

For example, when $X = \mathbb{CP}^2$ and $Y = S^3$, we can see $X^Y_{(0)} \simeq (\mathbb{CP}^2 \times K(Q,2))_{(0)}$ by the calculation in [2]. In particular, ${X_{(0)}}^Y$ is formal even though X does not have the rational homotopy type of a product of odd dimensional spheres. Also we must consider each connected component of X^Y in the general case.

In the following sections, our category is CDGA, that is, the objects are commutative differential graded algebras (cdga) over Q, and the morphisms are maps of differential graded algebra. Also, $H^*(\ )$ means $H^*(\ ;Q)$ and $I(S)$ denotes the ideal in the algebra A generated by a basis of a subspace S in A. When B is a subalgebra of A and both A and B contain S, then $I(S)$ denotes the ideal in the algebra A and $I_B(S)$ the ideal in the algebra B, unless otherwise noted.

2. Two changes of KS-basis

When a cdga $\mathcal{A}$ is formal, we can choose a minimal model $\mathcal{M} = (\wedge V, d)$ of $\mathcal{A}$ such that $V = \mathrm{Ker}(d_{|V}) \oplus \mathrm{Ker}(\psi_{|V})$ for a quasi-isomorphism $\psi: \mathcal{M} \to (H^*(\mathcal{A}), 0)$. Therefore, according to [3, Thm 4.1], $\mathcal{A}$ is formal iff there is a complement N to $\mathrm{Ker}(d_{|V})$, $V = \mathrm{Ker}(d_{|V}) \oplus N$, such that any d-cocycle of $I(N)$ is d-exact. We remark this '$\mathcal{M}$' must be a normal minimal model. Conversely, if $\mathcal{M} = (\wedge V, d)$ is a normal minimal model and formal, $H^*(\mathcal{M})$ is generated by $\mathrm{Ker}(d_{|V})$ as an algebra (see [9, Lemma 1.8]). Therefore for any quasi-isomorphism $\psi: \mathcal{M} \to (H^*(\mathcal{A}), 0)$, we have $V = \mathrm{Ker}(d_{|V}) \oplus \mathrm{Ker}(\psi_{|V})$.

Following [8, p. 5], we use the term "change of KS-basis" in this paper as follows. Suppose that

$$(B^*, d_B) \longrightarrow (B^* \otimes \wedge V, \delta) \longrightarrow (\wedge V, \bar{\delta})$$

is a KS-extension with KS-basis $\{v_i\}_{i \in I}$, *i.e.*, a well-ordered basis of V such that $i < j$ if $|v_i| < |v_j|$ for each $i, j \in I$ and $\delta(v_i) \in B^* \otimes \wedge V_{<i}$. Define a map of algebras $\phi: B^* \otimes \wedge V \to B^* \otimes \wedge V$ by setting

$$\phi_{|B} = \mathrm{id}_B \quad \text{and} \quad \phi(v_i) = v_i + \chi_i$$

on basis elements of V, where $\chi_i \in B^* \otimes \wedge V_{<i}$ (To be exact, this is different from the definition of "KS-change of basis" of [8, p. 5] since χ_i may not be contained in $B^+ \otimes \wedge V$.) Finally, define a new differential D on $B^* \otimes \wedge V$ by

$$D = \phi^{-1} \circ \delta \circ \phi.$$

Then we have an isomorphism of KS-extensions

$$\begin{tikzcd}
(B^*, d_B) \arrow[r, "\text{incl.}"] \arrow[d, "="] & (B^* \otimes \wedge V, D) \arrow[r, "\text{proj.}"] \arrow[d, "\phi"', "\cong"] & (\wedge V, \bar{D}) \arrow[d, "\bar{\phi}"', "\cong"] \\
(B^*, d_B) \arrow[r, "\text{incl.}"] & (B^* \otimes \wedge V, \delta) \arrow[r, "\text{proj.}"] & (\wedge V, \bar{\delta}),
\end{tikzcd} \tag{2.1}$$

where $D_{|B^*} = \delta_{|B^*} = d_B$.