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THE NAGAEV-GUIVARC'H METHOD VIA THE KELLER-LIVERANI THEOREM

BY LOÏC HERVÉ & FRANÇOISE PÈNE

Abstract. — The Nagaev-Guivarc'h method, via the perturbation operator theorem of Keller and Liverani, has been exploited in recent papers to establish limit theorems for unbounded functionals of strongly ergodic Markov chains. The main difficulty of this approach is to prove Taylor expansions for the dominating eigenvalue of the Fourier kernels. The paper outlines this method and extends it by stating a multidimensional local limit theorem, a one-dimensional Berry-Esseen theorem, a first-order Edgeworth expansion, and a multidimensional Berry-Esseen type theorem in the sense of the Prohorov metric. When applied to the exponentially $\mathbb{L}^2$-convergent Markov chains, to the v-geometrically ergodic Markov chains and to the iterative Lipschitz models, the three first above cited limit theorems hold under moment conditions similar, or close (up to $\varepsilon > 0$), to those of the i.i.d. case.

Résumé (*La méthode de Nagaev-Guivarc'h via le théorème de Keller-Liverani*)

La méthode de Nagaev-Guivarc'h, via le théorème de perturbation de Keller et Liverani, a été appliquée récemment en vu d'établir des théorèmes limites pour des fonctionnelles non bornées de chaînes de Markov fortement ergodiques. La difficulté principale dans cette approche est de démontrer des développements de Taylor pour la valeur propre perturbée de l'opérateur de Fourier. Dans ce travail, nous donnons une présentation générale de cette méthode, et nous l'étendons en démontrant un

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théorème limite local multidimensionnel, un théorème de Berry-Esseen unidimensionnel, un développement d'Edgeworth d'ordre 1, et enfin un théorème de Berry-Esseen multidimensionnel au sens de la distance de Prohorov. Nos applications concernent les chaînes de Markov $\mathbb{L}^2$-fortement ergodiques, v-géométriquement ergodiques, et les modèles itératifs. Pour ces exemples, les trois premiers théorèmes limites cités précédemment sont satisfaits sous des conditions de moment dont l'ordre est le même (parfois à $\varepsilon > 0$ près) que dans le cas indépendant.

1. Introduction, setting and notations

Let $(X_n)_n$ be a Markov chain with values in $(E, \mathscr{E})$, with transition probability Q and with stationary distribution π. Let ξ be a π-centered random variable with values in $\mathbb{R}^d$ (with $d \geq 1$). We are interested in probabilistic limit theorems for $(\xi(X_n))_n$ namely:

- central limit theorem (c.l.t.),
- rate of convergence in the central limit theorem: Berry Esseen type theorem,
- multidimensional local limit theorem,
- first-order Edgeworth expansion (when $d = 1$).

We want to establish these results under moment conditions on ξ as close as possible to those of the i.i.d. case (as usual i.i.d. is the short-hand for "independent and identically distributed"). Let us recall some facts about the case when $(Y_n)_n$ is a sequence of i.i.d. $\mathbb{R}^d$-valued random variables (r.v.) with null expectation. If $Y_1 \in \mathbb{L}^2$, we have the central limit theorem and, under some additional nonlattice type assumption, we have the local limit theorem. If $Y_1 \in \mathbb{L}^3$ and $d = 1$, we have the uniform Berry-Esseen theorem, and the first-order Edgeworth expansion (under the nonlattice assumption). All these results can be proved thanks to Fourier techniques. If $Y_1 \in \mathbb{L}^3$, $(Y_n)_n$ satisfies a multidimensional Berry-Esseen type theorem (in the sense of the Prohorov metric). The proof of this last result uses Fourier techniques and a truncation argument.

To get analogous results for Markov chains, we shall use and adapt the Nagaev-Guivarc'h method, introduced in [63, 64] and [36, 37] in the case $d = 1$. This method is based on Fourier techniques and on the usual perturbation operator theory applied to the Fourier kernels $Q(t)(x, dy) = e^{it\xi(y)}Q(x, dy)$ ($t \in \mathbb{R}$). The idea is that $\mathbb{E}\big[e^{it\sum_{k=1}^{n}\xi(X_k)}\big]$ is close enough to an expression of the form $\lambda(t)^n$, and the calculations are then similar to those of the i.i.d. case. Indeed, let us recall that, if $(Y_n)_n$ is a sequence of i.i.d. random variables, then we have $\mathbb{E}\big[e^{it\sum_{k=1}^{n}Y_k}\big] = \big(\mathbb{E}[e^{itY_1}]\big)^n$.

The Nagaev-Guivarc'h method, also called the spectral method, has been widely strengthened and extended, especially since the 80's with the contribution of Le Page [56], Rousseau-Egele [70], Milhaud and Raugi [62]. This is fully described by Hennion and the first author in [43], where other references are given. Roughly speaking, to operate the spectral method, one needs the following strong ergodicity assumption (specified below) w.r.t. some Banach space $\mathcal{B}$, namely: $Q^n \to \pi$ in the operator norm topology of $\mathcal{B}$. Under this assumption, the sequence $(\xi(X_n))_n$ then satisfies the usual distributional limit theorems provided that (Q, ξ) verifies some operator-moment conditions on $\mathcal{B}$. This method is especially efficient when $\mathcal{B}$ is a Banach algebra and ξ is in $\mathcal{B}$. Unfortunately, on the one hand, since Banach algebras are often composed of bounded functions, the condition $\xi \in \mathcal{B}$ implies that ξ must be bounded. On the other hand, usual models as v-geometrically ergodic Markov chains or iterative Lipschitz models (typically $E = \mathbb{R}^p$) are strongly ergodic w.r.t. some weighted supremum normed space or weighted Lipschitz-type space which are not Banach algebras, and the above mentioned operator-moment conditions then hold under very restrictive assumptions involving both Q and ξ. For instance, in these models, the usual spetral method cannot be efficiently applied to the sequence $(X_n)_n$ (i.e. $\xi(x) = x$); an explicit and typical counter-example will be presented in Section 3.

In recent works [14, 32, 34, 39, 44, 46, 47], a new procedure, based on the perturbation theorem of Keller-Liverani [54] (see also [5] p. 177), allows to get round the previous difficulty and to greatly improve the Nagaev-Guivarc'h method when applied to unbounded functionals ξ. Our work outlines this new approach, and presents the applications, namely: a multidimensional local limit theorem, a one-dimensional Berry Esseen theorem, a first-order Edgeworth expansion. We establish these results under hypotheses close to the i.i.d. case. We also establish a multidimensional Berry-Esseen type theorem in the sense of the Prohorov metric under hypotheses analogous to $Y_1 \in \mathbb{L}^m$ with $m = \max(3, \lfloor d/2 \rfloor + 1)$ instead of $Y_1 \in \mathbb{L}^3$. The reason is that, when adapting [53], we can use Yurinskii's smoothing inequality (valid for r.v. in $\mathbb{L}^m$) but we cannot adapt Yurinskii's truncation argument.

When the usual perturbation theorem is replaced with that of Keller-Liverani, the main difficulty consists in proving Taylor expansions for the dominating eigenvalue $\lambda(t)$ of the Fourier kernel $Q(t)$. This point is crucial here. Such expansions may be obtained as follows:

(A) To get Taylor expansion at $t = 0$, one can combine the spectral method with more probabilistic arguments such as martingale techniques [47]. In this paper, this method is just outlined: the local limit theorem obtained in [46] is extended to the multidimensional case, and the one-dimensional uniform Berry-Esseen theorem of [47] is here just recalled for completeness.

(B) To establish the others limit theorems, we shall use a stronger property: the regularity of the eigen-elements of $Q(\cdot)$ on a neighbourhood of $t = 0$. We shall see that this can be done by considering the action of $Q(t)$ on a "chain" of suitable Banach spaces instead of a single one as in the classical approach. This method, already used for other purposes in [33, 40, 58], has been introduced in the spectral method [44] to investigate the c.l.t. for iterative Lipschitz models. It is here specified and extended to general strongly ergodic Markov chains, and it will provide the one-dimensional Edgeworth expansion and the multidimensional Berry-Esseen type theorem.

Next, we introduce our probabilistic setting, and the functional notations and definitions, helpful in defining the operator-type procedures of the next sections.

Probabilistic setting. — $(X_n)_{n\geq 0}$ is a Markov chain with general state space $(E, \mathcal{E})$, transition probability Q, stationary distribution π, initial distribution μ, and $\xi = (\xi_1, \ldots, \xi_d)$ is a $\mathbb{R}^d$-valued π-integrable function on E such that $\pi(\xi) = 0$ (i.e. the ξ_i's are π-integrable and $\pi(\xi_i) = 0$). The associated random walk in $\mathbb{R}^d$ is denoted by

$$S_n = \sum_{k=1}^{n} \xi(X_k).$$

We denote by $|\cdot|_2$ and $\langle \cdot, \cdot \rangle$ the euclidean norm and the canonical scalar product on $\mathbb{R}^d$. For any $t \in \mathbb{R}^d$ and $x \in E$, we define the Fourier kernels of (Q, ξ) as

$$Q(t)(x, dy) = e^{i\langle t,\, \xi(y)\rangle} Q(x, dy).$$

$\mathcal{N}(0, \Gamma)$ denotes the centered normal distribution associated to a covariance matrix Γ, and "$\xrightarrow{\mathcal{D}}$" means "convergence in distribution". Although $(X_n)_{n\geq 0}$ is not a priori the canonical version, we shall slightly abuse notation and write $\mathbb{P}_\mu$, $\mathbb{E}_\mu$ to refer to the initial distribution. For any μ-integrable function f, we shall often write $\mu(f)$ for $\int f d\mu$. For $x \in E$, δ_x will stand for the Dirac mass: $\delta_x(f) = f(x)$. Finally, a set $A \in \mathcal{E}$ is said to be π-full if $\pi(A) = 1$, and Q-absorbing if $Q(a, A) = 1$ for all $a \in A$.

Functional setting. — Let $\mathcal{B}, X$ be complex Banach spaces. We denote by $\mathcal{L}(\mathcal{B}, X)$ the space of the bounded linear operators from $\mathcal{B}$ to X, and by $\|\cdot\|_{\mathcal{B},X}$ the associated operator norm, with the usual simplified notations $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{B}, \mathcal{B})$, $\mathcal{B}' = \mathcal{L}(\mathcal{B}, \mathbb{C})$, for which the associated norms are simply denoted by $\|\cdot\|_{\mathcal{B}}$. If $T \in \mathcal{L}(\mathcal{B})$, $r(T)$ denotes its spectral radius, and $r_{\text{ess}}(T)$ its essential spectral radius. For the next use of the notion of essential