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AEPPLI COHOMOLOGY CLASSES ASSOCIATED WITH GAUDUCHON METRICS ON COMPACT COMPLEX MANIFOLDS

by Dan Popovici

Abstract. — We propose the study of a Monge-Ampère-type equation in bidegree $(n-1, n-1)$ rather than $(1,1)$ on a compact complex manifold X of dimension n for which we prove ellipticity and uniqueness of the solution subject to positivity and normalization restrictions. Existence will hopefully be dealt with in future work. The aim is to construct a special Gauduchon metric uniquely associated with any Aeppli cohomology class of bidegree $(n-1, n-1)$ lying in the Gauduchon cone of X that we hereby introduce as a subset of the real Aeppli cohomology group of type $(n-1, n-1)$ and whose first properties we study. Two directions for applications of this new equation are envisaged: to moduli spaces of Calabi-Yau $\partial\bar\partial$-manifolds and to a further study of the deformation properties of the Gauduchon cone beyond those given in this paper.

Résumé (*Classes de cohomologie d'Aeppli associées à des métriques de Gauduchon sur les variétés complexes compactes*)

Nous proposons l'étude d'une équation de type Monge-Ampère en bidegré $(n-1, n-1)$ plutôt que $(1,1)$ sur une variété complexe compacte X de dimension n pour laquelle nous démontrons l'ellipticité et l'unicité des solutions soumises à des contraintes de positivité et de normalisation. Nous espérons que la question de l'existence pourra être traitée dans un travail ultérieur. Le but est de construire une métrique de Gauduchon spéciale associée de manière unique à une classe de cohomologie d'Aeppli quelconque de bidegré $(n-1, n-1)$ appartenant au cône de Gauduchon de X que nous introduisons comme un sous-ensemble du groupe de cohomologie d'Aeppli réel de type $(n-1, n-1)$ et dont nous étudions les premières

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propriétés. Des applications de cette nouvelle équation sont envisagées dans deux directions : aux espaces de modules de $\partial\bar{\partial}$-variétés de Calabi-Yau et à une étude des propriétés de déformations du cône de Gauduchon au-delà de celles décrites dans ce travail.

1. Introduction

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. The main theme of this paper is the interaction between various kinds of metrics (especially Gauduchon metrics) on X and certain cohomology theories (especially the Aeppli cohomology) often considered on X.

On the metric side, let $\omega > 0$ be a C^∞ positive definite $(1,1)$-form (i.e., a Hermitian metric) on X. The following diagram sums up the definitions of well-known kinds of Hermitian metrics and the implications among them.

$$
\begin{array}{lll}
d\omega = 0 & \Longrightarrow \exists\, \alpha^{0,2} \in C^\infty_{0,2}(X,\mathbb{C}) \text{ s.t.} & \Longrightarrow \partial\bar{\partial}\omega = 0 \\
 & \quad d(\overline{\alpha^{0,2}} + \omega + \alpha^{0,2}) = 0 & \\
(\omega \text{ is Kähler}) & (\omega \text{ is Hermitian-symplectic}) & (\omega \text{ is pluriclosed}) \\
\Downarrow & & \\
d\omega^{n-1} = 0 & \Longrightarrow \exists\, \Omega^{n-2,n} \in C^\infty_{n-2,n}(X,\mathbb{C}) \text{ s.t.} & \Longrightarrow \partial\bar{\partial}\omega^{n-1} = 0 \\
 & \quad d(\overline{\Omega^{n-2,n}} + \omega^{n-1} + \Omega^{n-2,n}) = 0 & \\
(\omega \text{ is balanced}) & (\omega \text{ is strongly Gauduchon (sG)}) & (\omega \text{ is Gauduchon}).
\end{array}
$$

Recall that of the above six types of metrics, only Gauduchon metrics always exist ([7]). Compact complex manifolds X carrying any of the other five types of metrics inherit the name of the metric (Kähler, balanced, etc).

The conditions on the top line in the above diagram bear on the metric in bidegree $(1,1)$, while those on the bottom line are their analogues in bidegree $(n-1,n-1)$. It is a well-known fact in linear algebra (see e.g., [11]) that for every smooth $(n-1,n-1)$-form $\Gamma > 0$ on X, there exists a unique smooth $(1,1)$-form $\gamma > 0$ on X such that $\gamma^{n-1} = \Gamma$. We denote it $\gamma = \Gamma^{\frac{1}{n-1}}$ and call γ the $(n-1)$st root of Γ. The power-root bijection between positive definite C^∞ forms of types $(1,1)$ and $(n-1,n-1)$ suggests a possible duality between the metric properties in these two bidegrees. The following observation (noticed before in [8] and references therein as a consequence of more general results) gives a further hint. We give below a quick proof.

Proposition 1.1. — *If ω is both pluriclosed and balanced, then ω is Kähler.*

Proof. — The pluriclosed assumption on ω translates to any of the following equivalent properties:

$$\partial\bar\partial\omega = 0 \iff \partial\omega \in \ker\bar\partial \iff \star(\partial\omega) \in \ker\partial^\star, \tag{1}$$

the last equivalence following from the well known formula $\partial^\star = -\star\bar\partial\star$, where $\star = \star_\omega : \Lambda^{p,q}T^\star X \to \Lambda^{n-q,n-p}T^\star X$ is the Hodge-star isomorphism defined by ω for arbitrary $p, q = 0, \dots, n$.

On the other hand, the balanced assumption on ω translates to any of the following equivalent properties:

$$d\omega^{n-1} = 0 \iff \partial\omega^{n-1} = 0 \iff \partial\omega \text{ is primitive}, \tag{2}$$

the last equivalence following from the formula $\partial\omega^{n-1} = (n-1)\,\omega^{n-2}\wedge\partial\omega$. Now, since $\partial\omega$ is primitive by (2), a well-known formula (valid for any Hermitian metric ω) given e.g., in [19, Proposition 6.29, p. 150] spells

$$\star(\partial\omega) = i\,\frac{\omega^{n-3}}{(n-3)!}\wedge\partial\omega = \frac{i}{(n-2)!}\,\partial\omega^{n-2}.$$

Since $\star(\partial\omega) \in \ker\partial^\star$ by (1), we get by applying $\partial^\star$ in the above identities:

$$\partial^\star\partial\omega^{n-2} = 0, \quad \text{so} \quad 0 = \langle\langle\partial^\star\partial\omega^{n-2}, \omega^{n-2}\rangle\rangle = ||\partial\omega^{n-2}||^2, \quad \text{so} \quad \partial\omega^{n-2} = 0.$$

Since $\partial\omega^{n-2} = (n-2)\,\omega^{n-3}\wedge\partial\omega$, the last identity above gives $\omega^{n-3}\wedge\partial\omega = 0$. Now, $\partial\omega$ is a form of degree 3 while the Lefschetz operator on 3-forms

$$L_\omega^{n-3} : \Lambda^3 T^\star X \longrightarrow \Lambda^{2n-3}T^\star X, \qquad \alpha \mapsto \omega^{n-3}\wedge\alpha,$$

is an isomorphism (see e.g., [19, Lemma 6.20, p. 146]—no assumption on the Hermitian metric ω is needed). It follows that $\partial\omega = 0$, i.e., ω is Kähler. □

On the cohomological side, recall that for all $p, q = 0, \dots, n$, the Bott-Chern cohomology group of type (p, q) is defined as

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) = \frac{\ker(d : C^\infty_{p,\,q}(X) \to C^\infty_{p+1,\,q}(X) + C^\infty_{p,q+1}(X))}{\operatorname{Im}(\partial\bar\partial : C^\infty_{p-1,\,q-1}(X) \to C^\infty_{p,\,q}(X))},$$

while the Aeppli cohomology group of type (p, q) is defined as

$$H^{p,\,q}_{A}(X,\,\mathbb{C}) = \frac{\ker(\partial\bar\partial : C^\infty_{p,\,q}(X) \to C^\infty_{p+1,\,q+1}(X))}{\operatorname{Im}(\partial : C^\infty_{p-1,\,q}(X) \to C^\infty_{p,\,q}(X)) + \operatorname{Im}(\bar\partial : C^\infty_{p,\,q-1}(X) \to C^\infty_{p,\,q}(X))}.$$

There are always well-defined linear maps from $H^{p,\,q}_{BC}(X,\,\mathbb{C})$, from $H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ (the Dolbeault cohomology group of type (p, q)) and from $H^{p+q}(X, \mathbb{C})$ (the De Rham cohomology group of degree $p+q$) to $H^{p,\,q}_{A}(X,\,\mathbb{C})$ but, in general, they are neither injective, nor surjective.

We will be often considering the case when X is a $\partial\bar\partial$-manifold. This means that the $\partial\bar\partial$-lemma holds on X, i.e., for all p, q and for any smooth d-closed