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DIFFERENTIAL FORMS IN POSITIVE CHARACTERISTIC AVOIDING RESOLUTION OF SINGULARITIES

by Annette Huber, Stefan Kebekus & Shane Kelly

Abstract. — This paper studies several notions of sheaves of differential forms that are better behaved on singular varieties than Kähler differentials. Our main focus lies on varieties that are defined over fields of positive characteristic. We identify two promising notions: the sheafification with respect to the cdh-topology, and right Kan extension from the subcategory of smooth varieties to the category of all varieties. Our main results are that both are cdh-sheaves and agree with Kähler differentials on smooth varieties. They agree on all varieties under weak resolution of singularities.

A number of examples highlight the difficulties that arise with torsion forms and with alternative candiates.

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1. Introduction

Sheaves of differential forms play a key role in many areas of algebraic and arithmetic geometry, including birational geometry and singularity theory. On singular schemes, however, their usefulness is limited by bad behavior such as the presence of torsion sections. There are a number of competing modifications of these sheaves, each generalizing one particular aspect. For a survey see the introduction of [14].

In this article we consider two modifications, Ω^n_{cdh} and Ω^n_{dvr}, to the presheaves Ω^n of relative k-differentials on the category $\mathsf{Sch}(k)$ of separated finite type k-schemes. By Ω^n_{cdh} we mean the sheafification of Ω^n with respect to the cdh-topology, cf. Definition 5.5, and by Ω^n_{dvr} we mean the right Kan extension along the inclusion $\mathsf{Sm}(k) \to \mathsf{Sch}(k)$ of the restriction of Ω^n to the category $\mathsf{Sm}(k)$ of smooth k-schemes, cf. Definition 5.2.

The following are three of our main results.

THEOREM 1.1. — *Let k be a perfect field and $n \geq 0$.*

1. *(Theorem 5.11). If X is a smooth k-variety then $\Omega^n(X) \cong \Omega^n_{\mathrm{cdh}}(X)$. The same is true in the rh- or eh-topology.*
2. *(Observation 5.3, Proposition 5.12). Ω^n_{dvr} is a* cdh*-sheaf and the canonical morphism*
$$\Omega^n_{\mathrm{cdh}} \to \Omega^n_{\mathrm{dvr}}$$
is a monomorphism.
3. *(Proposition 5.13). Under weak resolution of singularities, this canonical morphism is an isomorphism $\Omega^n_{\mathrm{cdh}} \cong \Omega^n_{\mathrm{dvr}}$.* □

Item 1 was already observed by Geisser, assuming a strong form of resolution of singularities, [8]. We are able to give a proof which does not assume any conjectures. The basic input into the proof is a fact about torsion forms (Theorem 5.8): given a torsion form on an integral variety, there is a blow-up such the pull-back of the form vanishes on the blow-up.

1.1. Comparison to known results in characteristic zero. — This paper aims to extend the results of [14] to positive characteristic, avoiding to assume resolution of singularities if possible. The following theorem summarizes the main results known in characteristic zero.

THEOREM 1.2 ([14]). — *Let k be a field of characteristic zero, X a separated finite type k-scheme, and $n \geq 0$.*

1. *The restriction of Ω^n_{h} to the small Zariski site of X is a torsion-free coherent sheaf of $\mathcal{O}_X$-modules.*
2. *If X is reduced we have*
$$\Omega^n(X)/\{\textit{torsion elements}\} \subseteq \Omega^n_{\mathrm{h}}(X)$$

and if X is Zariski-locally isomorphic to a normal crossings divisor in a smooth variety then

$$\Omega^n(X)/\{\textit{torsion elements}\} \cong \Omega^n_{\mathrm{h}}(X).$$

3. *If X is smooth, then $\Omega^n(X) \cong \Omega^n_{\mathrm{h}}(X)$ and $H^i_{Zar}(X, \Omega^n) \cong H^i_{\mathrm{h}}(X, \Omega^n_{\mathrm{h}})$ for all $i \geq 0$. The same is true using the* cdh-*or* eh-*topology in place of the* h-*topology.*
4. *We have $\Omega^n_{\mathrm{dvr}} \cong \Omega^n_h$, cf. Definition 5.2.* □

Failure of Items 1 and 2 in positive characteristic. — In positive characteristic, the first obstacle to this program one discovers is that $\Omega^n_{\mathrm{h}} = 0$ for $n \geq 1$, cf. Lemma 6.1. This is due to the fact that the geometric Frobenius is an h-cover, which induces the zero morphism on differentials. However, almost all of the results of [14] are already valid in the coarser cdh-topology, and remain valid in positive characteristic if one assumes that resolutions of singularities exist. So let us use the cdh-topology in place of the h-topology. But even then, Items 1 and 2 of Theorem 1.2 seem to be lost causes:

Corollary 1.3 (Corollary 5.16, Corollary 5.17, Example 3.6). — *For perfect fields of positive characteristic, there exist varieties X such that the restriction of Ω^1_{cdh} to the small Zariski site of X is not torsion-free.*

Moreover, there exist morphisms $Y \to X$ and torsion elements of $\Omega^1_{\mathrm{cdh}}(X)$ (resp. $\Omega^1(X)$) whose pull-back to $\Omega^1_{\mathrm{cdh}}(Y)$ (resp. $\Omega^1(Y)$) are not torsion. □

Note that functoriality of torsion forms over the complex numbers is true, cf. Theorem 3.3, [18, Corollary 2.7].

Positive results. — On the positive side, Item 1 in Theorem 1.1 can be seen as an analog of Item 3 in Theorem 1.2. In particular, we can give an unconditional statement of the case $i = 0$. In a similar vein, Items 2 and 3 of Theorem 1.1 relate to Item 4 in Theorem 1.2.

1.2. Other results. — Many of the properties of Ω^n_{dvr} hold for a more general class of presheaves, namely unramified presheaves, introduced by Morel, cf. Definition 4.5. The results mentioned above are based on the following very general result which should be of independent interest.

Proposition 1.4 (Proposition 4.18). — *Let S be a Noetherian scheme. If $\mathscr{F}$ is an unramified presheaf on $\mathsf{Sch}(S)$ then $\mathscr{F}_{\mathrm{dvr}}$ is an* rh-*sheaf. In particular, if $\mathscr{F}$ is an unramified Nisnevich (resp. étale) sheaf on $\mathsf{Sch}(S)$ then $\mathscr{F}_{\mathrm{dvr}}$ is a* cdh-*sheaf (resp.* eh-*sheaf).* □

In our effort to avoid assuming resolution of singularities, we investigated the possibility of a topology sitting between the cdh-and h-topologies which might allow the theorems of de Jong or Gabber on alterations to be used in place of resolution of singularities. An example of the successful application of such