MÉMOIRES DE LA SMF 101/102

STRICHARTZ ESTIMATES FOR SCHRÖDINGER EQUATIONS WITH VARIABLE COEFFICIENTS

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Société Mathématique de France 2005
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification*. — 35A17, 35A22, 35Q40, 35Q55.

***Key words and phrases*. —** Strichartz estimates, Schrödinger equations, dispersive estimates, FBI transform, Sjöstrand's theory.

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***Abstract*. —** We prove the (local in time) Strichartz estimates (for the full range of parameters given by the scaling unless the end point) for asymptotically flat and non trapping perturbations of the flat Laplacian in $\mathbb{R}^n$, $n \geqslant 2$. The main point of the proof, namely the dispersion estimate, is obtained in constructing a parametrix. The main tool for this construction is the use of the FBI transform.

Résumé **(Inégalités de Strichartz pour l'équation de Schrödinger à coefficients variables)**

On démontre les inégalités de Strichartz (locales en temps) pour l'ensemble des indices donnés par l'invariance d'échelle (sauf le point final) pour des perturbations asymptotiquement plates et non captantes du laplacien usuel de $\mathbb{R}^n$, $n \geqslant 2$. Le point principal de la preuve, à savoir l'estimation de dispersion, est obtenu en construisant une paramétrixe. L'outil principal de cette construction est la théorie de la transformation de FBI construite par Sjöstrand.

CONTENTS

1. Introduction and statement of the result 1

2. Preliminaries and reduction to the case of a small perturbation of the Laplacian 7

2.1. Preliminaries 7

2.2. Reduction to a small perturbation 10

3. Study of the flow 15

3.1. Preliminaries 15

3.2. The flow for short time 18

3.3. The forward flow from points in $\mathcal{S}_+$ and backward from $\mathcal{S}_-$ 19

3.4. Precisions on the flow in the general case 21

3.5. The flow from points in $(\mathcal{S}_+ \cap \mathcal{S}_-)^c$ 23

4. The phase equation 25

4.1. Statement of the result 25

4.2. The preparation theorem 27

4.3. The case of outgoing points 29

4.4. The case of incoming points 53

4.5. The phase for small θ 99

5. The transport equations 107

5.1. Statement of the result and preliminaries 107

5.2. The case of outgoing points 108

5.3. The case of incoming points 117

5.4. The amplitude for short time 145

6. Microlocal localizations and the use of the FBI transform 149

6.1. Preliminaries 149

6.2. The microlocalization procedure 151

6.3. The one sided parametrix 156

6.4. Conclusion of Chapter 6 169

7. The dispersion estimate and the end of the proof of Theorem 1.0.1 171
7.1. The dispersion estimate for the operators $K_{\pm}(t)$ 171
7.2. End of the proof of Theorem 2.2.1 192

Appendix 195
A.1. The Faa di Bruno Formula 195
A.2. Proof of Proposition 3.2.1 195
A.3. Proof of Proposition 3.3.2 196
A.4. Proof of Lemma 5.3.1 200

Bibliography 207

CHAPTER 1

INTRODUCTION AND STATEMENT OF THE RESULT

The purpose of this work is to provide a proof of the full (local in time) Strichartz estimates for the Schrödinger operator related to a non trapping asymptotically flat perturbation of the usual Laplacian in $\mathbb{R}^n$.

Let σ_0 be in $]0,1[$. We introduce a space of symbols which decay like $\langle x\rangle^{-1-\sigma_0}$ where $\langle x\rangle = (1+|x|^2)^{1/2}$. More precisely we set

(1.0.1)
$$\mathcal{B}_{\sigma_0} = \Big\{ a \in C^\infty(\mathbb{R}^n) : \forall\, \alpha \in \mathbb{N}^n,\ \exists\, C_\alpha > 0 : |\partial^\alpha a(x)| \leqslant \frac{C_\alpha}{\langle x\rangle^{1+|\alpha|+\sigma_0}}, \forall\, x \in \mathbb{R}^n \Big\}$$

Let P be a second order differential operator,

$$P = \sum_{j,k=1}^{n} D_j\big(g^{jk}(x)\, D_k\big) + \sum_{j=1}^{n} (D_j\, b_j(x) + b_j(x)\, D_j) + V(x), D_j = \frac{1}{i}\frac{\partial}{\partial x_j}\,, \tag{1.0.2}$$

with principal symbol $p(x,\xi) = \sum_{j,k=1}^n g^{jk}(x)\,\xi_j\,\xi_k$. (Here $g^{jk} = g^{kj}$).

We shall make the following assumptions.

$$\begin{cases} \text{(i)} & \text{The coefficients } g^{jk},\ b_j,\ V \text{ are real valued, } 1 \leqslant j \leqslant k \leqslant n. \\ \text{(ii)} & \text{There exists } \sigma_0 > 0 \text{ such that } g^{jk} - \delta_{jk} \in \mathcal{B}_{\sigma_0},\ b_j \in \mathcal{B}_{\sigma_0}. \\ & \text{Here } \delta_{jk} \text{ is the Kronecker symbol.} \\ \text{(iii)} & V \in L^\infty(\mathbb{R}^n). \end{cases} \tag{1.0.3}$$

(1.0.4) There exists $\nu > 0$ such that for every (x,ξ) in $\mathbb{R}^n \times \mathbb{R}^n$, $p(x,\xi) \geqslant \nu\,|\xi|^2$.

Then P has a self-adjoint extension with domain $H^2(\mathbb{R}^n)$.

Now we associate to the symbol p the bicharacteristic flow given by the following equations for $j = 1, \ldots, n$,

$$\begin{cases} \dot{x}_j(t) = \dfrac{\partial p}{\partial \xi_j}\,(x(t),\, \xi(t)), & x_j(0) = x_j, \\ \dot{\xi}_j(t) = -\dfrac{\partial p}{\partial x_j}\,(x(t),\, \xi(t)), & \xi_j(0) = \xi_j. \end{cases} \tag{1.0.5}$$

We shall denote by $(x(t,x,\xi), \xi(t,x,\xi))$ the solution, whenever it exists, of the system (1.0.5). In fact it is an easy consequence of (1.0.3) and (1.0.4) that this flow exists for all t in $\mathbb{R}$. Indeed by (1.0.4) we have

$$\nu\,|\xi(t)|^2 \leqslant p(x(t),\,\xi(t)) = p(x,\xi),$$

and it follows from (1.0.4) that

$$|\dot{x}_j(t)| \leqslant 2\sum_{k=1}^{n} |g^{jk}(x)\,\xi_k(t)| \leqslant C\,|\xi(t)| \leqslant C\,\nu^{-1/2}\,p(x,\xi)^{1/2}.$$

Our last assumption will be the following.

(1.0.6) For all (x,ξ) in $T^*\mathbb{R}^n \smallsetminus \{0\}$ we have $\lim_{t\to\pm\infty} |x(t,x,\xi)| = +\infty.$

This means that the flow is not trapped backward nor forward. Now let us denote by e^{-itP} the solution of the following initial value problem

$$\begin{cases} i\,\dfrac{\partial u}{\partial t} - Pu = 0 \\ u(0,\cdot) = u_0. \end{cases} \tag{1.0.7}$$

Then the main result of this work is the following.

Theorem 1.0.1. — *Assume that the operator P satisfies the conditions (1.0.3), (1.0.4), (1.0.6). Let $T > 0$ and (q,r) be a couple of real numbers such that $q > 2$ and $\frac{2}{q} = \frac{n}{2} - \frac{n}{r}$. Then there exists a positive constant C such that*

$$\|e^{-itP}u_0\|_{L^q([-T,T],L^r(\mathbb{R}^n))} \leqslant C\,\|u_0\|_{L^2(\mathbb{R}^n)}, \tag{1.0.8}$$

for all u_0 in $L^2(\mathbb{R}^n)$.

Such estimates are known in the literature under the name of Strichartz estimates. They have been proved for the flat Laplacian by Strichartz [**Str**] when $p = q = \frac{2n+4}{n}$ and extended to the full range of (p,q) given by the scaling by Ginibre-Velo [**GV**] and Yajima [**Y**]. The limit case $q = 2$ (the end point) when $n \geqslant 3$ is due to Keel-Tao [**KT**]. These estimates have been a key tool in the study of non linear equations. Very recently several works appeared showing a new interest for such estimates in the case of variable coefficients. Staffilani-Tataru [**ST**] proved Theorem 1.0.1 under conditions (1.0.4) and (1.0.6) for compactly supported perturbations of the flat Laplacian. In [**B**] Burq gave an alternative proof of this result using the work of Burq-Gérard-Tzvetkov [**BGT**]. In the same work Burq announced without proof that if you accept to replace in the right hand side of (1.0.8) the L^2 norm by an H^ε norm, for any small $\varepsilon > 0$, then you can weaken the decay hypotheses on the coefficients of P in the sense that you may replace in the definition (1.0.1) of $\mathcal{B}_{\sigma_0}$ the power $|\alpha| + 1 + \sigma_0$ by $|\alpha| + \sigma_0$. We have also to mention a recent work of Hassell-Tao-Wunsch [**HTW1**] who proved in dimension $n = 3$ a weaker form of our result corresponding to the case where $q = 4$, $r = 3$, under conditions similar to ours. Still more recently these three authors announced the same result as ours under hypotheses on the coefficients similar to ours (see [**HTW2**]).

It is also worthwhile to mention the work of Burq-Gérard-Tzvetkov who investigate the Strichartz estimates on compact Riemannian manifolds. In that case they show that such estimates hold with the L^2 norm replaced by the $H^{1/q}$ norm. In the same paper these authors show that the same result holds on $\mathbb{R}^n$ when the coefficients of their Laplacian (and its derivatives) are merely bounded. Let us note also that these estimates concern also the wave equation and many works have been devoted to this case. However we would like to emphasize that, due to the finite speed of propagation, the extension to the variable coefficients case appear to be much less technical (see [**SS**]).

Let us now give some ideas on the proof. It is by now well known that a proof of the Strichartz estimates can be done using a dispersion result, duality arguments and the Hardy-Littlewood-Sobolev lemma. This has been formulated as an abstract result in the paper [**KT**] as follows. Assume that for every $t \in \mathbb{R}$ we have an operator $U(t)$ which maps $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and satisfies,

$$\begin{cases} \text{(i)} \;\; \|U(t)\,f\|_{L^2(\mathbb{R}^n)} \leqslant C\,\|f\|_{L^2(\mathbb{R}^n)}, \;\; \forall\, t \in \mathbb{R}, \;\; C \text{ independent of } t, \\ \text{(ii)} \; \|U(s)(U(t))^*\,g\|_{L^\infty(\mathbb{R}^n)} \leqslant C\,|t-s|^{-n/2}\,\|g\|_{L^1(\mathbb{R}^n)}, \;\; t \neq s, \end{cases}$$

then the Strichartz estimates (1.0.5) hold for $U(t)$. It is not difficult to see that the serious estimate to be proved is (ii). In the case when $U(t) = e^{it\Delta_0}$ (the flat Laplacian) this estimate is obtained by the explicit formula giving the solution in term of the data u_0. In the variable coefficients case such a formula is of course out of hope and the better we can have is a parametrix. However due to strong technical difficulties (which we try to explain below) which seem to be serious we are not able to write a parametrix for e^{-itP} so we have to explain what we do instead. First of all let $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that $\varphi_0(x) = 1$ if $|x| \leqslant \frac{3}{2}$ and $\operatorname{supp}\varphi_0 \subset [-1,1]$. With a large $R > 0$ we write

$$e^{-itP}\,u_0(x) = \varphi_0\Big(\frac{x}{R}\Big)\,e^{-itP}\,u_0(x) + \Big(1 - \varphi_0\Big(\frac{x}{R}\Big)\Big)\,e^{-itP}\,u_0(x) = v + w.$$

It is not difficult to see that the Strichartz estimates for v will be ensured by the result of Staffilani-Tataru [**ST**] while the same estimate for w leads to consider an operator which is a small perturbation of the Laplacian (see Chapter 2).

Now it is not a surprise that microlocal analysis is strongly needed in our proof. So let $\xi_0 \in \mathbb{R}^n$, $|\xi_0| = 1$ be a fixed direction. Let $\chi_0 \in C^\infty(\mathbb{R})$, $\chi_0(s) = 1$ if $s \leqslant \frac{3}{4}$, $\chi_0(s) = 0$ if $s \geqslant 1$, $0 \leqslant \chi_0 \leqslant 1$ and let us set $\chi_+(x) = \chi_0\big(-x\cdot\xi_0/\delta_1\big)$, $\chi_-(x) = \chi_0\big(x\cdot\xi_0/\delta_1\big)$, $\delta_1 > 0$. We set $U_+(t) = \chi_+\,e^{-itP}$, $U_-(t) = \chi_-\,e^{-itP}$. Now since $\chi_+(x) + \chi_-(x) \geqslant 1$ for all x in $\mathbb{R}^n$ then Strichartz estimates separately for $U_+(t)$ and $U_-(t)$ will give the result. It is therefore sufficient to prove the estimate (ii) above for $U_+(s)\;(U_+(t))^* = \chi_+\,e^{i(s-t)P}\,\chi_+$ (and for $U_-(s)(U_-(t))^*$). In our proof we shall construct a parametrix for these operators.

Our construction relies heavily on the theory of FBI transform (see Sjöstrand [**Sj**] and Melin-Sjöstrand [**MS**]) viewed as a Fourier integral operator with complex phase. One of the reason of our choice is that in our former works on the analytic smoothing effect [**RZ2**] we have already done similar constructions (but only near the outgoing points: see below). Let us explain very roughly the main ideas. The standard FBI transform is given by

$$Tv(\alpha,\lambda) = c_n\, \lambda^{3n/4} \int_{\mathbb{R}^n} e^{i\lambda(y-\alpha_x)\cdot\alpha_\xi - \frac{\lambda}{2}|y-\alpha_x|^2 + \frac{\lambda}{2}|\alpha_\xi|^2}\, v(y)\, dy \tag{1.0.9}$$

where $\alpha = (\alpha_x, \alpha_\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and c_n is a positive constant.

Let us note that the phase can be written $i\lambda\varphi_0$ where $\varphi_0(y,\alpha) = \frac{i}{2}\,(y-(\alpha_x+i\alpha_\xi))^2$. Then T maps $L^2(\mathbb{R}^n)$ into the space $L^2(\mathbb{R}^{2n}, e^{-\lambda|\alpha_\xi|^2}\, d\alpha)$. The adjoint T^* of T is given by a similar formula (see (6.1.2)) and we have,

$$T^*T \text{ is the identity operator on } L^2(\mathbb{R}^n). \tag{1.0.10}$$

We embed the transform T into a continuous family of FBI transform

$$\begin{cases} T_\theta v(\alpha,\lambda) = \lambda^{3n/4} \displaystyle\int_{\mathbb{R}^n} e^{i\lambda\varphi(\theta,y,\alpha)} a(\theta,y,\alpha)\, v(y)\, dy \text{ with} \\ \varphi(0,y,\alpha) = \frac{1}{2}(y-(\alpha_x+i\alpha_\xi))^2, a(0,y,\alpha) = c_n. \end{cases} \tag{1.0.11}$$

Let us set $U(\theta,t,\alpha,\lambda) = T_\theta[K_\pm(t)\, u_0](\alpha,\lambda)$, where $K_\pm(t) = \chi_\pm\, e^{-itP}\, \chi_\pm$. Then it is shown that if φ satisfies the eikonal equation,

$$\Big[\frac{\partial\varphi}{\partial\theta} + p\Big(x, \frac{\partial\varphi}{\partial x}\Big)\Big](\theta,x,\alpha) = 0, \tag{1.0.12}$$

and if the symbol a satisfies appropriate transport equations then U is a solution of the following equation

$$\Big(\frac{\partial U}{\partial t} + \lambda\,\frac{\partial U}{\partial\theta}\Big)(\theta,t,\alpha,\lambda) \sim 0.$$

It follows that essentially we have, $U(\theta,t,\alpha,\lambda) = V(\theta-\lambda t,\alpha,\lambda)$. In particular this shows that $U(0,t,\alpha,\lambda) = U(-\lambda t, 0,\alpha,\lambda)$. Written in terms of the transformations T_θ this reads

$$T[K_\pm(t)u_0](\alpha,\lambda) = T_{-\lambda t}[\chi_\pm^2\, u_0](\alpha,\lambda).$$

Applying T^* to both members and using (1.0.10) we obtain

$$K_\pm(t)\, u_0(x) = T^*\{T_{-\lambda t}[\chi_\pm^2\, u_0](\cdot,\lambda)\}(t,x).$$

Thus we have expressed the solution in terms of the data through a Fourier integral operator with complex phase.

This short discussion shows that as usual the main point of the proof is to solve the eikonal and transport equations. Let us point out the main difficulties which occur in solving these equations. They are of three types: the bad behavior of the flow from incoming points and for large time, the global (in θ, x) character of all our constructions and the mixing of C^∞ coefficients and complex variables (coming from the non real character of our phase). Let us discuss each of them. First of all